PROBLEM OF THE WEEK
Solution of Problem No. 12 (Spring 2013 Series)

Problem:

Let ϕ denote the Euler totient; $\phi(n)$ is the number of integers $1 \leq r \leq n$ such that $(r, n) = 1$. Define $\xi(n)$ as the sum of those $\phi(n)$ integers. Show that for $n > 2$, $\xi(n) = n\phi(n)/2$.

**This problem was proposed by Steve Spindler, Chicago.

Solution: (by Bennett Marsh, Sophomore, Engineering, Purdue University)

If r is coprime to n, then so is $n - r$. For otherwise there would be some $d > 1$ such that $d \mid n$ and $d \mid (n - r)$. But then $d \mid (n - (n - r)) = r$, contradicting the fact that r and n are coprime. Thus we see that the numbers which are coprime to n come in pairs that sum to n (r and $n - r$ are distinct since $n/2$ is never coprime to n, if $n > 2$). There are exactly $\phi(n)/2$ such pairs, so their sum is $\xi(n) = n\phi(n)/2$.

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Ahtsham Ali, HD Harald Bensom (Germany), Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Janet Dant (Batavia, IL), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Lincoln James (HSE&Co. Chicago), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (PostDoc, UCSD), Kevin Pardede (Indonesia), Benjamin Philabaum (Indianapolis, IN), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago), Chris Willy (Adjunct faculty, George Washington Univ.)