PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2013 Series)

Problem:

Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

when a, b, c, d **are arbitrary positive numbers.**

Solution: (by Marco Biagini, Math Teacher, Lucca, Italy)

Let $P(a, b, c, d) \in \mathbb{R}^4$ and $S(P) : \mathbb{R}^4 \to \mathbb{R} \quad S(P) = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$. Since the denominator of any term of the sum is less than $a+b+c+d$ we have

$$S > \frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d} = 1$$

Considering that $\quad \forall \lambda > 0 \quad \frac{x}{y} < \frac{x+\lambda}{y+\lambda} \Leftrightarrow x < y \quad$ we also have

$$S < \frac{a+c}{a+b+c+d} + \frac{b+d}{a+b+c+d} + \frac{c+a}{a+b+c+d} + \frac{d+b}{a+b+c+d} = 2.$$

Now set $a = 1 \quad b = k \quad c = k^2 \quad d = k^2$ then

$$S(k) = \frac{1}{1+k+k^2} + \frac{k}{1+k+k^2} + \frac{k}{1+2k} + \frac{k^2}{1+2k^2} \to 1 \quad \text{as} \quad k \to 0$$

Changing set into $a = 1 \;\; b = k \;\; c = 1 \;\; d = k$ we get $S(k) = \frac{2}{1+2k} + \frac{2k}{2+k} \to 2 \quad \text{as} \quad k \to 0$ so there are points $A \in \mathbb{R}^4$ in which the value of S is arbitrarily close to 1 and points $B \in \mathbb{R}^4$ in which the value is arbitrarily close to 2. The restriction of S on any connected curve joining any pair of A and B is a continuous function, so by the intermediate value theorem we conclude that the range of S is the interval $(1, 2)$.

The problem was also solved by:

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