PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2013 Series)

Problem:

A random walk on the three dimensional integer lattice is defined as follows. The walker starts at $(0,0,0)$. A standard six sided die is rolled six times. After each roll the walker moves to one of its six nearest neighbors, according to the following protocol: if the die rolls 1, 2, 3, 4, 5, or 6 dots the walker jumps one unit in the $+x, -x, +y, -y, z, -z$ direction respectively.
Find the probability that after the sixth roll the walker is back at its starting point $(0,0,0)$.

Solution: (by Kilian Cooley, Senior, Math & AAE, Purdue University)

Denote by $P_n(i,j,k)$ the probability that the walker is at position (i,j,k) in the lattice after n moves. Then the rule governing the walker's motion implies that:

$$\begin{aligned} P_{n+1}(i,j,k) = &\frac{1}{6}(P_n(i+1,j,k) + P_n(i-1,j,k) \\ &+ P_n(i,j+1,k) + P_n(i,j-1,k) \\ &+ P_n(i,j,k+1) + P_n(i,j,k-1)) \end{aligned}$$

Define

$$Q_n(x,y,z) = \sum_{i,j,k=-n}^{n} P_n(i,j,k)x^i y^j z^k$$

With $Q_0 = 1$. Consequently,

$$Q_n(x,y,z) = \frac{1}{6^n}((x+x^{-1}) + (y+y^{-1}) + (z+z^{-1}))^n$$

$P_6(0,0,0)$ represents the probability taht the walker returns to the origin on the sixth step, and is the constant term in $Q_6(x,y,z)$. To find that term, expand Q_6 using the trinomial theorem as

$$Q_6(x,y,z) = \frac{1}{6^6} \sum_{n_1+n_2+n_3=6} \frac{6!}{n_1!n_2!n_3!}(x+x^{-1})^{n_1}(y+y^{-1})^{n_2}(z+z^{-1})^{n_3}$$

where n_1, n_2, n_3 are nonnegative integers. Each summand contributes to the constant term only if n_1, n_2, n_3 are all even, in which case the binomial theorem implies that the constant term is $\binom{n_1}{n_1/2}\binom{n_2}{n_2/2}\binom{n_3}{n_3/2}$. Equivalently,

$$\begin{aligned}
Q_6(x,y,z) =& \frac{6!}{6^6} \sum_{m_1+m_2+m_3=3} \frac{(x+x^{-1})^{2m_1}(y+y^{-1})^{2m_2}(z+z^{-1})^{2m_3}}{(2m_1)!(2m_2)!(2m_3)!} \\
P_6(0,0,0) =& \frac{6!}{6^6} \sum_{m_1+m_2+m_3=3} \frac{1}{(2m_1)!(2m_2)!(2m_3)!}\binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3} \\
=& \frac{6!}{6^6} \sum_{m_1+m_2+m_3=3} \frac{1}{(m_1!)^2(m_2!)^2(m_3!)^2}.
\end{aligned}$$

Calculating this sum shows

$$P_6(0,0,0) = \frac{6!}{6^6}\cdot\frac{31}{12} = \frac{155}{3888} \approx 0.03987.$$

The problem was also solved by:

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