PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2013 Series)

Problem:

Find the maximum possible value of $\prod_{i=1}^{n} a_i$ **given** $0 \le a_i \le i$**,** $\sum_{i=1}^{n} a_i = 3n$**.**

Solution: (by Sorin Rubinstein, TAU faculty,Tel Aviv, Israel)

Let $H_n = ([0,1] \times [0,2] \times \cdots \times [0,n]) \bigcap \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 3n\}$ and $\varphi : H_n \to R$, $\varphi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i$. Since $1 + 2 + \cdots + n = \frac{n(n+1)}{2} \ge 3n$ iff $n \ge 5$ it follows that H_n is void for $n = 1, 2, 3, 4$ and consists of the single point: $(1, 2, 3, 4, 5)$ if $n = 5$. Assume that $n \ge 6$. Since H_n is compact, the continuous function φ attains its maximal value on some point of H_n. Let $a = (a_1, a_2, \ldots, a_n)$ be such a point. Clearly, φ is not identical 0. Hence $a_i > 0, i = 1, 2, \ldots, n$. Let $s \in \{1, 2, 3\}$. Assume that $a_s < s$. Since $\sum_{i=1}^{n} a_i = 3n$ and $a_s < s$ it follows that there exist some $k \in \{s+1, s+2, \ldots, n\}$ such that $a_k > s$.

Define a point $a' = (a'_1, a'_2, \ldots, a'_n)$ by: $a'_i = \begin{cases} a_i, & i \ne s, k \\ s, & i = s \\ a_k + a_s - s, & i = k \end{cases}$. It follows that $\sum_{i=1}^{n} a'_i = \sum_{i=1}^{n} a_i = 3n$. Moreover, $0 < a_k + a_s - s < a_k$ implies that $a'_k < k$ and, consequently that $a' \in H_n$. On the other hand, from $a_s(a_k - s) < s(a_k - s)$ follows $a_s a_k < s(a_k + a_s - s) = a'_s a'_k$ and, consequently $\prod_{i=1}^{n} a_i < \prod_{i=1}^{n} a'_n$. This contradicts the choice of a. Thus $a_s = s$, $s = 1, 2, 3$. Then $a_4 + a_5 + \cdots + a_n = 3n - 6$. From the AM-GM inequality it follows that $\prod_{i=4}^{n} a_i \le \left(\frac{3n-6}{n-3}\right)^{n-3}$ with equality for $a_i = \frac{3n-6}{n-3}$, $i \ge 4$. Moreover since for $n \ge 6$, $\frac{3n-6}{n-3} \le 4$ it follows that $\left(1, 2, 3, \frac{3n-6}{n-3}, \ldots, \frac{3n-6}{n-3}\right) \in H_n$. Thus, the maximal value of $\prod_{i=1}^{n} a_i$ is $6 \cdot \left(\frac{3n-6}{n-3}\right)^{n-3}$ if $n \ge 6$ and 120 if $n = 5$.

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Three submitters assumed that the a_i had to be integers, and if they solved this equivalently hard problem that solution was counted correct.

Undergraduates: Kilian Cooley (Sr. Math & AAE) Bennett Marsh (So. Engr.), Rustam Orazaliyev (Fr. Actuarial Sci), Chenkai Wang (So. Math)

Graduates: Krishnaraj Sambath (ChE), Tairan Yuwen (Chemistry)

Others: Marco Biagini (Italy), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Jon Cohen (Grad Student, UMCP), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Peter Kornya (Retired Faculty, Ivy Tech), Chris Kyriazis (HS Teacher, Chalki, Greece), Steven Landy (Physics Faculty, IUPUI), Hai Viet Le (Student, Univ. of Texas, San Antonio), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math Teacher, Zionsville Community HS), Steve Spindler (Chicago)