PROBLEM OF THE WEEK
Solution of Problem No. 4 (Spring 2013 Series)

Problem:

Show there do not exist four integers x_1, x_2, x_3, x_4, not all zero, such that

$$x_1^2 + x_2^2 + x_3^2 - 7x_4^2 = 0$$

Solution: (by Steven Landy, Physics Faculty, IUPUI)

Show there do not exist four integers a, b, c, d, not all zero, such that

$$a^2 + b^2 + c^2 - 7d^2 = 0 \tag{1}$$

Let's rewrite (1) as

$$a^2 + b^2 + c^2 + d^2 = 8d^2 \tag{2}$$

If a, b, c, and d were all even and satisfied (2), we could cancel a factor of 4 from both sides making a "smaller" solution (unless the terms were all zero, which has been rejected). We could continue this until we came to a solution where at least one of a, b, c, d were odd. Now the squares of the integers mod 8 are 0, 1, and 4. So there is no way for the left hand side of (2) to be congruent to 0 mod 8 if one of the terms is odd.

The problem was also solved by:

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