PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2013 Series)

Problem:
Find the radius of convergence of the MacLaurin expansion of

$$f(x) = \int_0^\infty \frac{\mathbf{d}t}{e^t + xt}.$$

Solution: (by Chenkai Wang, Sophomore, Mathematics, Purdue University)

Claim : $\qquad R(f, 0) = e.$

Proof. First, fix t and expand inner function $\dfrac{1}{e^t + xt}$, we have

$$f(x) = \int_0^\infty \frac{1}{e^t + xt}\,\mathrm{d}t = \int_0^\infty \sum_{n=0}^\infty (-1)^n \frac{t^n}{e^{(t(n+1))}} x^n\,\mathrm{d}t. \tag{1}$$

The radius of convergence of the inner series is $\dfrac{e^t}{t}$ and since $\inf\limits_{t \geq 0} \dfrac{e^t}{t} = e$, this step is justified for $|x| < e$.

Next, fix x and let $S_{x,N}(t) = \sum\limits_{n=0}^{N} (-1)^n \dfrac{t^n}{e^{(t(n+1))}} x^n$ be the partial sum of the inner series. Then the integral becomes

$$\int_0^\infty \lim_{N\to\infty} S_{x,N}(t)\,\mathrm{d}t. \tag{2}$$

Because $S_{x,N}(t)$ is alternating, and the terms decrease in magnitude, $|S_{x,N}(t)|$ is uniformly bounded by the norm of its first term. By Dominated Convergence Theorem, we can interchange the infinite integral with the limit, then we have

$$\begin{aligned} f(x) = \int_0^\infty \lim_{N\to\infty} S_{x,N}(t)\,\mathrm{d}t &= \lim_{N\to\infty} \int_0^\infty \sum_{n=0}^{N} (-1)^n \frac{t^n}{e^{(t(n+1))}} x^n\,\mathrm{d}t \quad \text{(DCT)} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \int_0^\infty (-1)^n \frac{t^n}{e^{(t(n+1))}} x^n\,\mathrm{d}t \quad \text{(finite sum)} \\ &= \sum_{n=0}^\infty (-1)^n \frac{n!}{(n+1)^{n+1}} x^n \quad \text{(use integration by parts and induction).} \end{aligned} \tag{3}$$

Then apply Cauchy–Hadamard theorem to calculate the radius of convergence,

$$\frac{1}{R(f,0)} = \limsup_{n\to\infty} \left| (-1)^n \frac{n!}{(n+1)^{n+1}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{n!}{(n+1)^{n+1}} \right|^{\frac{1}{n}}. \tag{4}$$

By Stirling's formula the last limit is seen to be $1/e$, and we conclude that $R(f,0) = e$ as claimed.

The problem was also solved by:

Graduates: Krishnaraj Sambath (ChE), Tairan Yuwen (Chemistry)

Others: Nadir Amaioua (Graduate Student, Ecole Polytechnique, Canada), Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Jingmin Chen (Graduate Student, Drexel Univ.), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Ayush Gupta (Student, IIT, Delhi, India), Anastasios Kotronis (Athens, Greece), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Craig Schroeder (Postdoc. UCLA), Mehdi Sonthonnax (Quantitative Analyst, NY), Bharath Swaminathan (Caterpillar, India), Bjorn Vermeersch (Postdoc, Purdue Univ.)