PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2013 Series)

Problem:

Background: The solution to POW 6 of last semester (online) shows that

$$\Gamma(r) = \left\{ \sum_{k\in S} r^k : S \textbf{ is a subset of the positive integers} \right\}$$

is not an interval if $0 < r < 1/2$ **and is an interval if** $1/2 \leq r < 1$**.**

Find a number r**,** $-1 < r < 0$**, such that** $\Gamma(r)$ **is an interval and find a number** s**,** $-1 < s < 0$**, such that** $\Gamma(s)$ **is not an interval.**

Solution: (by Tairan Yuwen, Graduate Student, Chemistry, Purdue University)

For r in the range of $(-1,0)$, the largest number in $\Gamma(r)$ is obtained only if all r^{2n} terms are included (n is positive integer), which is $\dfrac{r^2}{1-r^2}$, while the smallest number in $\Gamma(r)$ is obtained only if all r^{2n-1} terms are included (n is positive integer), which is $\dfrac{r}{1-r^2}$. In order to let $\Gamma(r)$ form an interval, all numbers in the range of $\left[\dfrac{r}{1-r^2}, \dfrac{r^2}{1-r^2}\right]$, should be included in it.

According to previous conclusion from POW 6 in last semester, for any number $-1 < r < 0$, if $r^2 \geq 1/2$ then $\Gamma(r)$ should form an interval. Since when considering numbers only composed of r^{2n-1} terms, they cover the whole interval of $\left[\dfrac{r}{1-r^2}, 0\right]$, and when considering numbers only compose of r^{2n} terms, they cover the whole interval of $\left[0, \dfrac{r^2}{1-r^2}\right]$, therefore $\Gamma(r)$ covers the whole interval of $\left[\dfrac{r}{1-r^2}, \dfrac{r^2}{1-r^2}\right]$, and $r = -0.9$ is such an example.

The number s such that $\Gamma(s)$ does not form an interval could possibly be found among those numbers not satisfying $s^2 \geq 1/2$, and one example is $s = -0.1$. When considering all numbers in $\Gamma(s)$ that do not include the term of s^1, only numbers in the range of $\left[-\dfrac{0.001}{0.99}, \dfrac{0.01}{0.99}\right]$ could potentially be sampled based on previous discussion, such that even if the term $s^1 (= -0.1)$ is added back, $\Gamma(s)$ at most include all numbers in the range of $\left[-\dfrac{0.1}{0.99}, -\dfrac{0.089}{0.99}\right] \bigcup \left[-\dfrac{0.001}{0.99}, -\dfrac{0.01}{0.99}\right]$, with all numbers in the range of $\left(-\dfrac{0.089}{0.99}, -\dfrac{0.001}{0.99}\right)$ missing, therefore $\Gamma(s)$ is not an interval.

The problem was also solved by:

Undergraduates: Bennett Marsh (So. Engr.)

Others: Marco Biagini (Italy), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Oliver Kroll (Stanford Law School), Steven Landy (Physics Faculty, IUPUI), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Steve Spindler (Chicago)