PROBLEM OF THE WEEK

Solution of Problem No. 10(Spring 2014 Series)

Problem:

Let f be a positive and continuous function on the real line which satisfies $f(x+1) = f(x)$ for all numbers x.

Prove $\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx \geq 1$**.**

Solution # 1: (by Tairan Yuwen, Graduate Student, Chemistry, Purdue University)

Proof:

$$\begin{aligned}
\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx &= \int_0^{\frac{1}{2}} \frac{f(x)}{f(x+\frac{1}{2})}\,dx + \int_{\frac{1}{2}}^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx \\
&= \int_0^{\frac{1}{2}} \frac{f(x)}{f(x+\frac{1}{2})}\,dx + \int_0^{\frac{1}{2}} \frac{f(x+\frac{1}{2})}{f(x+1)}\,dx \\
&= \int_0^{\frac{1}{2}} \left[\frac{f(x)}{f(x+\frac{1}{2})} + \frac{f(x+\frac{1}{2})}{f(x+1)}\right] dx \\
&= \int_0^{\frac{1}{2}} \left[\frac{f(x)}{f(x+\frac{1}{2})} + \frac{f(x+\frac{1}{2})}{f(x)}\right] dx \\
&\geq \int_0^{\frac{1}{2}} 2 \times \left[\sqrt{\frac{f(x)}{f(x+\frac{1}{2})} \cdot \frac{f(x+(\frac{1}{2})}{f(x)}}\right] dx \\
&= 2\int_0^{\frac{1}{2}} dx = 1
\end{aligned}$$

Solution # 2: (by Gruian Cornel, Cluj-Napoca, Romania)

Actually for any positive integer n, $I_n = \int_0^1 \frac{f(x)}{f\left(x+\frac{1}{n}\right)} \geq 1$. Clearly $I_1 = 1$.

For any $n \geq 2$,

$$I_n = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{f(x)}{f\left(x+\frac{1}{n}\right)}\, dx = \sum_{k=0}^{n-1} \int_0^{\frac{1}{n}} \frac{f\left(x+\frac{k}{n}\right)}{f\left(x+\frac{k+1}{n}\right)}\, dx$$

$$= \int_0^{\frac{1}{n}} \left(\sum_{k=0}^{n-1} \frac{f\left(x+\frac{k}{n}\right)}{f\left(x+\frac{k+1}{n}\right)}\right) dx$$

Now apply AM–GM inequality and so

$$\sum_{k=0}^{n-1} \frac{f\left(x+\frac{k}{n}\right)}{f\left(x+\frac{k+1}{n}\right)} \geq n\left(\frac{f(x)}{f\left(x+\frac{1}{n}\right)} \cdot \frac{f\left(x+\frac{1}{n}\right)}{f\left(x+\frac{2}{n}\right)} \cdot \ldots \cdot \frac{f\left(x+\frac{n-1}{n}\right)}{f(x+1)}\right)^{\frac{1}{n}} = n.$$

Therefore $I_n \geq \int_0^{\frac{1}{n}} n\, dx = 1$.

Solution # 3: (by Francois Seguin, Amiens, France)

If f is 1 periodic continuous, and strictly positive and $\lambda \in \mathbb{R}$ then

$$\int_0^1 \frac{f(x+\lambda)}{f(x)}\, dx = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{f\left(\frac{k}{n}+\lambda\right)}{f\left(\frac{k}{n}\right)}$$

$$\geq \lim_{n\to+\infty} \sqrt[n]{\prod_{k=0}^{n-1} \frac{f\left(\frac{k}{n}+\lambda\right)}{f\left(\frac{k}{n}\right)}} = \lim_{n\to+\infty} \frac{e^{\frac{1}{n}\sum_{k=0}^{n-1} \ln\left(f\left(\frac{k}{n}+\lambda\right)\right)}}{e^{\frac{1}{n}\sum_{k=0}^{n-1} \ln\left(f\left(\frac{k}{n}\right)\right)}}$$

$$= e^{\int_0^1 \ln(f(x+\lambda))dx - \int_0^1 \ln(f(x))dx}$$

But as f is 1 periodic $\int_0^1 \ln(f(x+\lambda))dx = \int_\lambda^{1+\lambda} \ln(f(x))dx = \int_0^1 \ln(f(x))dx$. Hence $\int_0^1 \frac{f(x+\lambda)}{f(x)}dx \geq 1$.

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