PROBLEM OF THE WEEK
Solution of Problem No. 13(Spring 2014 Series)

Problem:

An airplane flies at constant airspeed c directly above a closed polygonal path in a plane, completing one circuit. Show that, compared to no wind, the presence of a wind of constant speed $k < c$ and constant direction will increase the time required.

Solution: (by Tin Lam, Engineer, St. Louis, MO)

Suppose each side of the closed polygonal path is of distance s_i and that $\ell = \sum_i s_i$ is the length of 1 circuit. Let $\vec{w}$ be the wind vector, with $|\vec{w}| = k$. Let $\vec{v_i}$ be the velocity vector of the plane on the i-th side of the closed polygonal path with $|\vec{v_i}| = c$. The ground velocity vector $\vec{g_i}$ is given by $\vec{g_i} = \vec{v_i} + \vec{w}$. Let $d_i = |\vec{g_i}|$ and $\hat{g}_i$ be the unit vector with the same direction as $\vec{g_i}$. Then, $\vec{g_i} = d_i \hat{g}_i$. Therefore, we have $\vec{v_i} + \vec{w} = \vec{g_i} = d_i \hat{g}_i$, or $\vec{v_i} = d_i \hat{g}_i - \vec{w}$. If we take the dot product with itself, we have:

$$c^2 = \vec{v}_i \cdot \vec{v}_i = d_i^2 - 2d_i \vec{w} \cdot \hat{g}_i + |\vec{w}|^2 = d_i^2 - 2d_i \vec{w} \cdot \hat{g}_i + k^2.$$

We have a quadratic in d_i, namely, $d_i^2 - d_i(2\vec{w} \cdot \hat{g}_i) + (k^2 - c^2) = 0$. Using the quadratic formula, we have:

$$d_i = \vec{w} \cdot \hat{g}_i \pm \sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}.$$

Since $c > k$, we can ignore the case where $\pm$ is negative as $d_i < 0$. Since d_i is the ground speed of the plane, we have that:

$$\begin{aligned}
t_{\text{wind}} &= \sum_i \frac{s_i}{\vec{w} \cdot \hat{g}_i + \sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}} = \sum_i \frac{s_i \vec{w} \cdot \hat{g}_i - s_i \sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}}{(\vec{w} \cdot \hat{g}_i)^2 - ((\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2))} \\
&= \sum_i \frac{s_i \vec{w} \cdot \hat{g}_i - s_i \sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}}{k^2 - c^2} = \sum_i \frac{s_i \vec{w} \cdot \hat{g}_i}{k^2 - c^2} + \sum_i \frac{s_i \sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}}{c^2 - k^2} \\
&\geq \frac{1}{k^2 - c^2} \sum_i s_i \vec{w} \cdot \hat{g}_i + \sum_i \frac{s_i \sqrt{c^2 - k^2}}{c^2 - k^2} = \frac{1}{k^2 - c^2} \sum_i s_i \vec{w} \cdot \hat{g}_i + \frac{1}{\sqrt{c^2 - k^2}} \sum_i s_i.
\end{aligned}$$

Note that $\sum_i s_i \vec{w} \cdot \hat{g}_i = k \sum_i s_i \cos\theta$ where θ is the angle between each side (as a vector) and $\vec{w}$. However, since the direction of $\vec{w}$ is constant, this is just the $\vec{w}$-component of the vector path, and since it is closed, we know $\sum_i s_i \cos\theta = 0$.

We have

$$t_{\text{wind}} \geq \frac{1}{\sqrt{c^2-k^2}} \sum_i s_i = \frac{\ell}{\sqrt{c^2-k^2}} > \frac{\ell}{c} = t_{\text{no wind}}, \text{ when } k > 0.$$

The problem was also solved by:

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