PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2014 Series)

Problem:

It is known that, for any positive integer m,

$$\lim_{n\to\infty} \sum_{\{k\geq 0: km\leq n\}} \binom{n}{km} \Big/ \sum_{j=0}^{n} \binom{n}{j} = \frac{1}{m}.$$

Prove this for $m = 2$.

<u>Solution 1: (by Yucheng Chen, College of Engineering, Purdue University)</u>

According to binomial theorem:

$$\begin{aligned} 2^n =&(1+1)^n = \sum_{i=0}^{n} \binom{n}{i}, \\ 0 =&(1-1)^n = \sum_{i=0}^{n} (-1)^n \binom{n}{i}, \end{aligned}$$

Therefore,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1},$$

$k = 2$,

$$\lim_{n\to\infty} \frac{\sum_{(k\geq 0: km\leq n)} \binom{n}{km}}{\sum_{j=0}^{n} \binom{n}{j}} = \lim_{n\to\infty} \frac{\sum_{k=0}^{2\left[\frac{n}{2}\right]} \binom{n}{2k}}{\sum_{j=0}^{n} \binom{n}{j}} = \lim_{n\to\infty} \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

<u>Solution 2: (by Steven Landy, Physics Faculty, IUPUI)</u>

The case $m = 2$ is the familiar fact that in each row of Pascal's triangle the even terms and odd terms have the same sums. We will show the general case. First we note that the restriction $km \leq n$, is unnecessary since the binomial coefficient is zero if it is not true.

The denominator is

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n \tag{1}$$

To calculate the numerator, let $\theta = 2\pi/m$, and let $\alpha = e^{i\theta}$. Now consider the binomials $(1+\alpha)^n, (1+\alpha^2)^n, \ldots (1+\alpha^m)^n$. When these are expanded in binomial coefficients and then added, only the terms with coefficients of the form $\binom{n}{km}$ survive. In fact we find

$$\sum_{\{k\geq 0\}} \binom{n}{km} = \frac{1}{m}\left((1+\alpha)^n + (1+\alpha^2)^n + \cdots + (1+\alpha^m)^n\right) \tag{2}$$

The final term in (2) is $\dfrac{2^n}{m}$. Each of the other $m-1$ terms is complex or zero. The term with the largest modulus is the first, for which the modulus is

$$\frac{1}{m}|1+e^{i\theta}|^n = \frac{2^n}{m}\left|\left[\cos\left(\frac{\theta}{2}\right)\right]^n\right| \tag{3}$$

Using (1), (2) and (3) we find that

$$\sum_{\{k\geq 0\}} \binom{n}{km} \Big/ \sum_{j=0}^{n} \binom{n}{j} = \frac{1}{m} + q \quad \text{where} \quad q \leq (m-1)\left|\left[\cos\left(\frac{\theta}{2}\right)\right]^n\right|.$$

We can see that q goes to zero for large n, giving $x = \dfrac{1}{m}$.

The problem was also solved by:

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