PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2014 Series)

Problem:

Let p be a polynomial in the variables $x_1, x_2, \ldots, x_n$. Show that if there is a number C such that $|p(x_1, \ldots, x_n)| \leq C$ for all real $x_1, x_2, \ldots, x_n$ then there is a number r such that $p(x_1, \ldots, x_n) = r$ for all $x_1, \ldots, x_n$.

Solution 1: (by Bennett Marsh, Physics/Math. Junior, Purdue University)

Assume that $|p(\vec{x})| \leq C$ for all $\vec{x} \in \mathbb{R}^n$, and that there exist $\vec{x}_1, \vec{x}_2$ such that $p(\vec{x}_1) \neq p(\vec{x}_2)$. Define $f(t) = p(\vec{x}_1 + (\vec{x}_2 - \vec{x}_1)t)$. Now $f(t)$ is a polynomial in t, and since $f(0) \neq f(1)$, it is nonconstant, say of degree $m > 0$. Then letting $f(t) = \sum_{k=0}^{m} a_k t^k$, we see that $\lim_{t\to\infty} f(t)/t^m = a_m \neq 0$. But this implies that $\lim_{t\to\infty} f(t) = \pm\infty$, so $f(t)$, and thus also $p(\vec{x})$, is unbounded. This contradicts the initial assumption, so $p(\vec{x})$ must in fact be constant.

Solution 2: (by Craig Schroeder, UCLA Postdoc)

Fix $x_1, \ldots, x_n$ and let $f(t) = p(x_1 t, \ldots, x_n t)$. Then f is a univariate polynomial which is bounded and so $f(1) = f(0)$, i.e. $p(x_1, x_2, \ldots, x_n) = p(0, 0, \ldots, 0)$. [Then use that bounded polynomials in one variable are bounded].

The problem was also solved by:

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Others: Shin-ichiro Seki (Graduate Student, Osaka University); Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Esmaeil Parsa (Math. Lecturer, Iran) Perfetti Paolo (Roma, Italy), M. Rajeswari (TA, India), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), David Stoner (HS Student, Aiken, S. Carolina),