PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2014 Series)

Problem:

Let $a_1, a_2, \ldots, a_n$ be positive numbers.

Find the smallest possible value of $\sum_{k=1}^{n} \frac{a_k}{a_{i_k}}$, where $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$.

<u>Solution 1: (by Yucheng Chen, First Year Engineering, Purdue University)</u>

According to Arithmetic Mean–Geometric Mean inequality,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

The equality holds if and only if $x_1 = x_2 = \cdots = x_n$

$$\frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{a_{ik}} \geq \sqrt[n]{\frac{a_1}{a_{i1}} \frac{a_2}{a_{i2}} \cdots \frac{a_n}{a_{in}}}.$$

Since $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$

$$\frac{a_1}{a_{i1}} \frac{a_2}{a_{i2}} \cdots \frac{a_n}{a_{in}} = 1$$

$$\sum_{k=1}^{n} \frac{a_k}{a_{ik}} \geq n.$$

The equality holds if and only if $a_k = a_{ik}$, so the smallest possible value is n.

<u>Solution 2: (by Manuel Gutierrez, Junior, EE, Purdue University)</u>

Order of the sum won't matter, so order the terms so that the numerators are increasing. Note every numerator must be used as a demoninator. $b_1, b_2, \ldots, b_n$ is an ordered list of positive numbers from $a_1, a_2, \ldots, a_n$.

Let $c_1, c_2, \ldots, c_n = b_{i1}, b_{i2}, \ldots, b_{in}$, an arbitrary permutation of $b_1, b_2, \ldots, b_n$. Assume $\sum_{k=1}^{n} \frac{b_k}{c_k}$ is then the minimum, but note that swapping two demoninators c_i and c_j, where $i < j$, $c_i > c_j$, and $b_i < b_j$ will reduce the sum further since the sum of those two terms is now $\frac{b_i}{c_j} + \frac{b_j}{c_i}$, and $\frac{b_i}{c_j} + \frac{b_j}{c_i} < \frac{b_i}{c_i} + \frac{b_j}{c_j}$.

Since $i < j \Rightarrow b_i < b_j \Rightarrow b_i(c_i - c_j) < b_j(c_i - c_j)$, where c_i and c_j are positive.

$$\Rightarrow b_i c_i - b_i c_j < b_j c_i - b_j c_j \Rightarrow b_i c_i + b_j c_j < b_j c_i + b_i c_j$$

$$\Rightarrow \frac{b_i c_i + b_j c_j}{c_i c_j} < \frac{b_j c_i + b_i c_j}{c_i c_j} \Rightarrow \frac{b_i}{c_j} + \frac{b_j}{c_i} < \frac{b_i}{c_i} + \frac{b_j}{c_j}.$$

Continuing to make these swaps in the list of denominators eventually leads to an ordered list of denominators equal to $b_1, b_2, \ldots, b_n$. Then no more such swaps can be made and the minimum sum is reached.

$$\sum_{k=1}^{n} \frac{b_k}{c_k} = \sum_{k=1}^{n} \frac{b_k}{b_k} = n$$

The problem was also solved by:

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