PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2014 Series)

Problem:

Where do you place five points on the unit circle of the plane to maximize the edge length of the inscribed simple polygon with these points as vertices if two of the five points are $(1,0)$ and $(-1,0)$?

<u>Solution: (by Yucheng Chen, First Year Engineering, Purdue University)</u>

Since two of the points are $(-1,0)$ and $(1,0)$, three points left are needed to be placed in the unit circle. If 3 points are all in the same side of the x-axis, suppose they are all above the x-axis. Suppose central angles of edges are $\theta_1, \theta_2, \theta_3, \theta_4$, we have $\theta_1+\theta_2+\theta_3+\theta_4=\pi$. Edge $r_i = 2\sin\dfrac{\theta_i}{2}$ Circumference of the polygon $C = 2\left(\sin\dfrac{\theta_1}{2}+\sin\dfrac{\theta_2}{2}+\sin\dfrac{\theta_3}{2}+\sin\dfrac{\theta_4}{2}\right)$ Since $f(x)=\sin x$ is concave down in $\left[0,\dfrac{\pi}{2}\right]$, according to Jensen's inequality, $C \le 4\times 2\sin\left(\dfrac{\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}}{4}\right)$, the equality holds if and only if $\theta_1=\theta_2=\theta_3=\theta_4$. Therefore, the maximum circumference of polygon in this situation is $2+8\sin\dfrac{\pi}{8}$. If 1 point is on one side of x-axis and 2 points are on another, suppose 1 point is above x-axis and 2 are below it.

Above the x-axis, suppose central angles of edges are θ_1, θ_2, we have $\theta_1+\theta_2=\pi$. Edge $r_i = 2\sin\dfrac{\theta_i}{2}$

$$C_1 = 2\left(\sin\frac{\theta_1}{2}+\sin\frac{\theta_2}{2}\right)$$
$$C_1 \le 2\times 2\sin\frac{\pi}{4} = 2\sqrt{2}.$$

Likewise, $C_2 \le 3\times 2\sin\frac{\pi}{6} = 3$. Therefore, the maximum circumference of polygon in this situation is $3+2\sqrt{2}$. Since $2+8\sin\frac{\pi}{8} < 3+2\sqrt{2}$, the maximum length of the incribed polygon is $3+2\sqrt{2}$.

The problem was also solved by:

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