PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2015 Series)

Problem:
Let a be the number with a ternary expansion .1001100011**. Show that a is the sum of two positive numbers which both have ternary expansions which use only zeros and twos. (These ternary expansions may have an infinite number of digits.)**

[Remark: One way to work this problem, not the easiest way, is to show that any number in [0,1] is the sum of two such numbers.]

Solution by Tin Lam, Engineer, St. Louis, MO

First we will prove that for all $t \in \left[0, \frac{1}{2}\right]$, we can find $x, y \geq 0$ where $t = x + y$, and x, y contain only zeroes and ones. Let $t = 0.t_1t_2t_3\ldots$, $x = 0.x_1x_2x_3\ldots$, and $y = 0.y_1y_2y_3\ldots$ be their ternary representations. For each i, let $x_i = y_i = 0$ if $t_i = 0$. Let $x_i = y_i = 1$ if $t_i = 2$, and lastly, if $t_i = 1$, let $x_i = 1$ and $y_i = 0$. Note that for possible digit of t_i, we can assign $x_i, y_i \in \{0, 1\}$. Then, for all $a = 2t \in [0, 1]$, there exist $x, y > 0$ where $2x + 2y = a$, and that $2x$ and $2y$ contain only zeroes and twos.

Solution: $a = 0.1001100011_3 = \frac{20659}{59049}$, then $\frac{a}{2} = 0.01120111201\overline{1}_3$. We can decompose this into $0.01120111201\overline{1}_3 = 0.01110111101\overline{1}_3 + 0.000100001_3$. Lastly, we have that:

$$0.1001100011_3 = 0.02220222202\overline{2}_3 + 0.000200002_3.$$

The problem was also solved by:

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