PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2015 Series)

Problem:

Let $x_1 \in (0,1)$. Iteratively define intervals $I_1, I_2, \ldots$ and points $x_2, x_3, \ldots$ by: I_k is the longest sub-interval of $(0,1)$ not containing any of the points $x_i, 1 \le i \le k$, (If ties choose the leftmost interval.) and x_k is the midpoint of I_{k-1}. Prove there is a number N not depending on x_1, such that the length of I_n does not exceed $\frac{20}{n}$ if $n \ge N$.)

*****The constant 20 appearing in the problem is not best possible, as was discovered by all the solvers. We thank Matthew Lim for pointing out an error in the previously posted solution. Lim's solution shows what interval lengths actually occur and makes it possible to show that the best constant is 2. Below is an edited version of Lim's solution.

Solution by Matthew Lim

Let $|I|$ be the length of interval I. We show that for any $x_1 \in (0,1)$ and all $n \ge 1$, $|I_n| < \frac{4}{n}$. Note that the leftmost criterion for ties does not affect the sequence of interval lengths.

1. Let $A = \max(x_1, 1-x_1)$, $B = \min(x_1, 1-x_1)$, so $A \ge B > 0$ and $A + B = 1$. Let $m = \max\left\{k \in \mathbb{N} : 2^{k-1} \le \frac{A}{B}\right\}$, so $2^{m-1} \le \frac{A}{B} < 2^m$. It follows that $\frac{A}{2^m} < B \le \frac{A}{2^{m-1}}$ and that $\frac{A}{2^{m+k}} < \frac{B}{2^k} \le \frac{A}{2^{m+k-1}} < \frac{1}{2^{m+k-1}}$ for $k \ge 0$.
2. Thus, in stage 1, $|I_1| = A$, and all bisections occur in I_1, producing 2^k intervals of length $\frac{A}{2^k}$ for each $0 \le k \le m-1$ followed by one of length B. In stage 2, for each $k \ge 0$, there are 2^k intervals of length $\frac{B}{2^k}$ followed by 2^{m+k} intervals of length $\frac{A}{2^{m+k}}$, or 2^{m+k} intervals of length $\frac{A}{2^{m+k}}$ followed by 2^{k+1} of length $\frac{B}{2^{k+1}}$.
3. We now determine the indices for the various intervals in each stage.
   (a) In stage 1, $|I_n| = \frac{A}{2^k}$ for $2^k \le n < 2^{k+1}$. Thus $n|I_n| < \frac{2^{k+1}A}{2^k} < 2$.
   (b) In the first part of stage 2, $|I_n| = \frac{B}{2^k}$ for
   $$2^k + 2^{m+k} - 1 \le n < 2^{k+1} + 2^{m+k} - 1.$$
   Thus $n|I_n| < \left(2^{k+1} + 2^{m+k}\right)\frac{B}{2^k} = (2 + 2^m)B \le \left(2 + 2^m\right)\frac{A}{2^{m-1}} \le 4.$
   In the second part of stage 2, $|I_n| = \frac{A}{2^{m+k}}$

for $2^{k+1}+2^{m+k}-1 \leq n < 2^{k+1}+2^{m+k+1}-1$.

Thus $n|I_n| < \left(2^{k+1}+2^{m+k+1}\right)\dfrac{A}{2^{m+k}} = \left(\dfrac{1}{2^{m-1}}+2\right)A \leq 3$.

We see that for all $n \geq 1$, $|I_n| < \dfrac{4}{n} < \dfrac{20}{n}$, as desired.

The problem was also solved by:

Others: Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Aaron Hassan (Sydney, Australia), Craig Schroeder (Postdoc. UCLA), Jiazhen Tan (HS Student, China), Tairan Yuwen (Postdoc, Chemistry, Purdue U)