PROBLEM OF THE WEEK
Solution of Problem No. 12 (Spring 2015 Series)

Problem:
What is the maximum value of $\sum_{i<j} |P_i - P_j|^2$ over all possible choices of $2n$ points on the ellipsoid $x^2 + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$.
Remark: It is shown in POW problem 5, Fall 2009 (solution online) that for the analogous problem for the unit sphere the maximum value is attained if and only if the centroid of the P_i is the origin.

<u>**Solution by Bennett Marsh, Senior. Physics & Math, Purdue University**</u>

We have

$$\begin{aligned}\sum_{i<j} |P_i - P_j|^2 &= \frac{1}{2}\sum_{i,j=1}^{2n} (|P_i|^2 + |P_j|^2 - 2P_i \cdot P_j) \\ &= \frac{1}{2}\left[2n\sum_i |P_i|^2 + 2n\sum_j |P_j|^2 - 2\sum_{ij} P_i \cdot P_j\right] \\ &= 2n\sum_i |P_i|^2 - \sum_i P_i \cdot \sum_j P_j \\ &= 2n\sum_{i=1}^{2n} |P_i|^2 - \left|\sum_{i=1}^{2n} P_i\right|^2.\end{aligned}$$

The first term is clearly maximized when all points P_i have the maximum distance from the origin of 3. The second term is minimized when $\sum P_i = 0$. We can satisfy both of these simultaneously if n points are chosen to be $(0, 0, 3)$ and the other n points are chosen to be $(0, 0, -3)$. In this case,

$$\sum_{i<j} |P_i - P_j|^2 = 2n \cdot 2n \cdot 3^2 - 0 = 36n^2.$$

The problem was also solved by:

Undergraduates: Victor Lee (Fr. CS), Jiaqi Zhou (Math)

Others: Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Zhiwei Fang (Tianjin University of Finance and Economics, China), Matthew Lim, Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Tairan Yuwen (Postdoc, Chemistry, Purdue U)