”

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2015 Series)

Problem:

Show that if A is a subset of the real numbers R and if f is a function from A to R such that there is a constant c for which $|f(x) - f(y)| = c|x - y|$ if x and y are in A, then there is a function g from R to R such that $g(x) = f(x)$ if x is in A and $|g(x) - g(y)| = c|x - y|$ for all numbers x and y in R.

This problem was contributed by Michael M. Brady of Houston, Texas.

Solution by Wei-Xiang Lien, Taiwan

Assume the subset A contains at least two numbers. By $|f(x) - f(y)| = c|x - y|$, we know $c \geq 0$. If $c = 0$, then $f(x) = f(y) = k$ for any $x, y \in A$, for some constant $k \in \mathbf{R}$. Taking $g(x) \equiv k$ on $\mathbf{R}$ is enough. Otherwise, $c \neq 0$, we may consider:

(a) If A contains exactly two numbers, say s, t, and $s > t$, then

$|f(s) - f(t)| = c|s - t| = c(s - t),\ f(s) - f(t) = c(s - t)$ or $-c(s - t)$.

Taking the linear function $g(x) = f(t) + (x - t) \cdot \dfrac{f(s) - f(t)}{s - t}$ on $\mathbf{R}$ is enough.

(b) If A contains more then two numbers, then for any three numbers $s, t, u \in A$, where $s > t > u$, we have

$$\frac{f(s) - f(t)}{s - t} = c \quad \text{or} \quad -c \tag{1}$$
$$\frac{f(t) - f(u)}{t - u} = c \quad \text{or} \quad -c \tag{2}$$
$$\frac{f(s) - f(u)}{s - u} = c \quad \text{or} \quad -c. \tag{3}$$

Before going on, we need a property here as

[Property 1]: If a, b, c, d are real numbers and b, d are positive, then

$$\min\left(\frac{a}{b}, \frac{c}{d}\right) \leq \frac{a + c}{b + d} \leq \max\left(\frac{a}{b}, \frac{c}{d}\right).$$

[Proof]: Let $s = \min\left(\frac{a}{b}, \frac{c}{d}\right)$, then $a \geq bs, c \geq ds$. So $a + c \geq (b + d)s$, and $\dfrac{a + c}{b + d} \geq s = \min\left(\frac{a}{b}, \frac{c}{d}\right)$. The remaining part of the proof is similar. If $\frac{a}{b} \neq \frac{c}{d}$, both inequalities are strict.

Now we continue (b) by (1), (2) as if $\frac{f(s)-f(t)}{s-t} \neq \frac{f(t)-f(u)}{t-u} = \frac{f(u)-f(t)}{u-t}$, then

$\left\{\frac{f(s)-f(t)}{s-t}, \frac{f(t)-f(u)}{t-u}\right\} = \{c, -c\}$.

By Property 1, $-c < \frac{f(s)-f(u)}{s-u} = \frac{f(s)-f(t)+f(t)-f(u)}{s-t+t-u} < c$, which contradicts to (3).

So $\frac{f(s)-f(t)}{s-t} = \frac{f(u)-f(t)}{u-t}$ for any three numbers $s, t, u \in A$. So $\frac{f(x)-f(y)}{x-y}$ is a constant (c or $-c$) for any $x, y \in A$. Take $g(x) = f(t) + (x-t) \cdot \frac{f(s)-f(t)}{s-t}$ on $\mathbf{R}$.

The problem was also solved by:

Undergraduates: Bennett Marsh (Sr. Physics & Math)

Graduates: Tairan Yuwen (Chemistry)

Others: Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Steven Landy (Physics Faculty, IUPUI), Matthew Lim, Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA)