PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2015 Series)

Problem:
Let w be the greatest common divisor of m and n. Show that $3^w - 1$ is the greatest common divisor of $3^m - 1$ and $3^n - 1$.

Solution by Craig Schroeder, Postdoc, UCLA

If $m = n$, the statement is trivial.
If $m = 0$, then $w = \gcd(0, n) = n$ and $\gcd(3^m - 1, 3^n - 1) = \gcd(0, 3^n - 1) = 3^n - 1 = 3^w - 1$.
The case $n = 0$ is similar.
Finally, consider the case $n > m > 0$ ($m > n > 0$ is similar). Note that

$$\begin{aligned}\gcd(3^m - 1, 3^n - 1) &= \gcd(3^m - 1, (3^n - 1) - (3^m - 1)) = \gcd(3^m - 1, 3^n - 3^m) \\ &= \gcd(3^m - 1, (3^{n-m} - 1)3^m) = \gcd(3^m - 1, 3^{n-m} - 1)\end{aligned}$$

and

$$\gcd(m, n) = \gcd(m, n - m).$$

This process can be repeated until $m = n$, at which point the conclusion is obvious.

The problem was also solved by:

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Others: KD Harald Bensom (Germany), M.V. Channakeshava (Retired Teacher, Bangalore), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Sandipan Dey (UMBC Alumni), Tom Engelsman (Tampa, FL), Rick Shilling & Bruce Grayson (Orlando, FL), Chung–Chin Jian (Postdoc, National Taiwan U), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Wei–Xiang Lien (Miaoli, Taiwan), Matthew Lim, Adem Limani (Student, U of Lund, Sweden), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Luciano Santos (Teacher, Portugal), Mehtaab Sawhney (HS Student, Commack HS, NY), George–Petru Scarlatescu (Student, Pitesti, Romania), Mehdi Sonthonnax (New York), David Stoner (HS Student, Aiken, S. Carolina), Feng Tian (Student, Duke U), Tairan Yuwen (Postdoc, Chemistry, Purdue U)