PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2015 Series)

Problem:

Let f be a continuous strictly decreasing concave (down) function on $[0,1]$ which is twice differentiable on $(0,1)$ and satisfies $f(0)=1$ and $f(1)=0$. Find, with proof, the point(s) on $\{(x,y,z): x+y+z=1, \quad x,y,z\geq 0\}$ where $f(x)f(y)f(z)$ is largest.

Solution by Luciano dos Santos, Teacher, Lisboa, Portugal

As f is concave, we can apply the Jensen's inequality: $f\left(\dfrac{x+y+z}{3}\right)\geq \dfrac{f(x)+f(y)+f(z)}{3}$. As $x+y+z=1$, $f\left(\dfrac{1}{3}\right)\geq \dfrac{f(x)+f(y)+f(z)}{3}$. The equality holds if $x=y=z$.

But $\dfrac{f(x)+f(y)+f(z)}{3}\geq \sqrt[3]{f(x)f(y)f(z)}$ (AM-GM inequality). The equality holds if [and only if] $f(x)=f(y)=f(z)$. So the maximum value of $f(x)f(y)f(z)$ is $f\left(\dfrac{1}{3}\right)^3$ (when $x=y=z=\dfrac{1}{3}$).

The problem was also solved by:

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Others: Charles Burnette (Graduate Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Sandipan Dey (UMBC Alumni), Mohammed Hamami (AT & T), Chung–Chin Jian (Postdoc, National Taiwan U), Steven Landy (Physics Faculty, IUPUI), Wei–Xiang Lien (Miaoli, Taiwan), Matthew Lim, Vaibhav Panvalkar, Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), George–Petru Scarlatescu (Student, Pitesti, Romania), Craig Schroeder (Postdoc. UCLA), David Stoner (HS Student, Aiken, S. Carolina), Jiazhen Tan (HS Student, China), William Wu (Quantitative Engineering Design Inc.), Tairan Yuwen (Postdoc, Chemistry, Purdue U)