## PROBLEM OF THE WEEK Solution of Problem No. 9 (Spring 2015 Series)

## **Problem:**

Let  $D = \{(i, j) : i \text{ is an integer and } j \text{ is either 0, 1, or 2}\}$ . Find a function h on D such that for all i, h(i, 0) = h(i, 2) = 0, h(0, 1) = 1, and h(i, 1) = (1/4)[h(i-1, 1) + h(i+1, 1) + h(i, 0) + h(i, 2)].

## Solution by Fedeli Mohammed Elwardi, Student, Mhamed Bouguerra University, Algeria

Let  $D = \{(i, j) : i \text{ is an integer}, j \text{ is either } 1, 2 \text{ or } 3\}$ . We look for a function h verifying these conditions: for all i,

$$h(i,0) = h(i,2) = 0, \quad h(0,1) = 1$$
 (A)

and

$$h(i,1) = (1/4)[h(i-1,1) + h(i+1,1) + h(i,0) + h(i,2)].$$
(B)

For the conditions (A) it's easy to think about taking for example:

$$h(i,j) = j(2-j)a_i$$

where  $a_i$  is a sequence of reals with  $a_0 = 1$ . Finding  $a_i$ : the (B) becomes now:

$$4h(i,1) = h(i-1,1) + h(i+1,1).$$

After substituting *i* by i + 1, we get 4h(i + 1, 1) = h(i, 1) + h(i + 2, 1) or  $4a_{i+1} = a_{i+2} + a_i$ .

This set of sequences contain at least a geometric one, let us assume that

$$a_i = cb^i$$
 (c, b are reals).

Then

$$c(b^i)(b^2 - 4b + 1) = 0.$$

Or  $b = 2 \pm \sqrt{3}$ . This proves that the sequences  $a_i = c_1(2-\sqrt{3})^i + c_2(2+\sqrt{3})^i$  verifies (B). The condition  $a_0 = 1$  implies that  $c_2 = 1 - c_1$ . Now we can say that the function h defined on D by

$$h(i,j) = j(2-j) \left[ \mu (2-\sqrt{3})^i + (1-\mu)(2+\sqrt{3})^i \right]$$

where  $\mu$  is a constant real is a solution of this problem.

## The problem was also solved by:

<u>Undergraduates</u>: Adam Kline (Fr. Chem/Phys), Victor Lee (Fr. CS), Bennett Marsh (Sr. Physics & Math), Rustam Orazaliyev (Sr. Actuarial Sci), Jack VanSchaik (Fr. Math)

<u>Others</u>: KD Harald Bensom (Germany), Hailu Bikila (Turkey), K. Bradifield (PhD Student, Michigan State U), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Sandipan Dey (UMBC Alumni), Sean Dillard (Duke U), Matthew Dunlap (Student, U of Akron), Tom Engelsman (Tampa, FL), Rick Shilling & Bruce Grayson (Orlando, FL), Aaron Hassan (Sydney, Australia), Kipp Johnson (Valley Catholic HS teacher, Oregon), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Wei–Xiang Lien (Miaoli, Taiwan), Matthew Lim, Matthew McMullen (Instructor, Otterbein Univ, OH), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Mehtaab Sawhney (HS Student, Commack HS, NY), Craig Schroeder (Postdoc. UCLA), Luis Silva (Portugal), Jason L. Smith (Professor, Richland Community College, IL), Jiazhen Tan (HS Student, China), Feng Tian (Student, Duke U), William Wu (Quantitative Engineering Design Inc.), Tairan Yuwen (Postdoc, Chemistry, Purdue U)