Piecewise Geometry

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Main objects

Our project investigates lattice polygons and polytopes, and the ways in which they can be cut into smaller pieces while preserving the lattice structure. Based on Peter Greenberg's paper, we study a special piecewise linear homeomorphism generated by maps in $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. However, unlike the planar case, having the same volume in three dimensions does not guarantee scissors congruence for polytopes.

Definition

1. SL_2Z is a group with underlying set $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$.
An affine extension of a vector space V by a linear group G is typically the group of transformations that includes both:
Linear operations from G (rotations, shears, scaling).
Translations by vectors in V.
This is mathematically formalized as the semi-direct product: $V \rtimes G$.
An element of this group is a pair (v, γ) where $v \in V$ and $\gamma \in G$. It acts on a vector x via the formula: $(v, \gamma) \cdot x = \gamma x + v$
2. We will denote by $\frac{1}{N}Z$ the subgroup of Q generated by $\frac{1}{N}$, and by A_N (resp. A_0) the affine extension of $\frac{1}{N}Z^2$ (resp. Q^2) generated by SL_2Z.
A rational line (resp. integral line) is a line passing through two rational(resp. integral) points in the plane.

Definition: pZ_N Homeomorphism

Let U and V be open subsets of the plane $\mathbb{R}^2$. An orientation-preserving homeomorphism $g : U \to V$ is called a pZ_N **homeomorphism** if there exists a finite set of rational lines $\mathcal{L} = \{l_1, \ldots, l_k\}$ such that:

1. The domain U is partitioned into connected components by these lines. Let

$$U' = U \setminus \bigcup_i l_i.$$

2. For every connected component C of U', the restriction of g to C acts as an element of the affine group A_N. That is, for each component C, there exists some $\gamma_C \in A_N$ such that:

$$g(x) = \gamma_C \cdot x \quad \text{for all } x \in C.$$

Proposition

If P is a polygon, the area $a(P)$of $intP$ is invariant under pZ_0 maps. There is also an invariant "length."
There is a function $L : \mathscr{C} \to \mathbb{Q}$ which takes positive values, such that

(a) **(invariance)** If $P \in \mathscr{C}$, $P \subseteq U$, and $g : U \to V$ is a pZ_0-homeomorphism, then $L(P) = L(g(P))$.

(b) **(subdivision)** $L(\overline{v_0 \cdots v_n}) = L(\overline{v_0 \cdots v_k}) + L(\overline{v_k \cdots v_n})$.

(c) **(homothety)** $L(NP) = NL(P)$, $P \in \mathscr{C}$, where NP is the image of P under the map $(x, y) \mapsto (Nx, Ny)$.

(d) **(no metric)** If $a, b \in \mathbb{Q}^2$, then $\inf(\overline{av_1 \cdots v_{n-1}b}) = 0$.

Proposition (Pick)

Let P be an integral polygon. The number of points of $\mathbb{Z}^2$ in $\operatorname{int} P$ is

$$a(P) - \frac{1}{2}\#(P \cap \mathbb{Z}^2) + 1.$$

(Note that $\#(P \cap \mathbb{Z}^2) = L(P)$)

Definition

An *N-segment* $\overline{ab}$ is a segment so that $\overline{ab} \cap \frac{1}{N}\mathbb{Z}^2 = \{a, b\}$.
An *N-triangle* is a triangle $\overline{abca}$ whose sides are N-segments, and whose interior contains no points of $\frac{1}{N}\mathbb{Z}^2$.
If $P = \overline{v_0 \cdots v_n v_0}$ is a polygon and $v_i \in \frac{1}{N}\mathbb{Z}^2$, then an *$N$-triangulation* of P is a triangulation by N-triangles.

Lemma

Let T_1 and T_2 be 1-triangles, with vertices $a_i \in T_i$. There is a unique element $g \in A_1$ such that $gT_1 = T_2$ and $ga_1 = a_2$.

Definition

Let P and Q be integral polygons. Then $f : \operatorname{int} \overline{P} \to \operatorname{int} \overline{Q}$ is a *1-triangulated homeomorphism* if

(i) f is simple,

(ii) f is a composite of 1-triangulated homeomorphisms, or

(iii) $\operatorname{int} \overline{P} = \operatorname{int} \overline{P_1} \cup \operatorname{int} \overline{P_2}$,
$\operatorname{int} \overline{Q} = \operatorname{int} \overline{Q_1} \cup \operatorname{int} \overline{Q_2}$, where P_i and Q_i are integral polygons,

$$\operatorname{int} \overline{P_1} \cap \operatorname{int} \overline{P_2} = \overline{v_0 \cdots v_n},$$

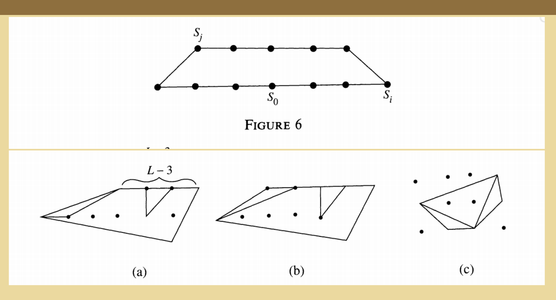
$\operatorname{int} \overline{Q_1} \cap \operatorname{int} \overline{Q_2} = \overline{w_0 \cdots w_n}$, with $\overline{v_i v_{i+1}}$ and $\overline{w_i w_{i+1}}$ 1-segments, and $f_i : \operatorname{int} \overline{P_i} \to \operatorname{int} \overline{Q_i}$, $i = 1, 2$, are 1-triangulated homeomorphisms such that $f_i(v_j) = w_j$.

Then, defining $f : \operatorname{int} \overline{P} \to \operatorname{int} \overline{Q}$ by setting $f|_{P_i} \equiv f_i$, f is a 1-triangulated homeomorphism.

Theorem: Greenberg's Theorem

Let P and Q be integral polygons of equal area and length, and let $p \in P \cap \mathbb{Z}^2$ and $q \in Q \cap \mathbb{Z}^2$. Then there exists a 1-triangulated homeomorphism $f : \operatorname{int} \overline{P} \to \operatorname{int} \overline{Q}$, with $f(p) = q$.

Figure



Definition (Ehrhart Polynomial)

Let $P \subset \mathbb{R}^d$ be a lattice polytope, for each positive integer k, consider the dilation kP. The *Ehrhart function* of P is defined as

$$i_P(k) = \#(kP \cap \mathbb{Z}^d),$$

Ehrhart's theorem states that $i_P(k)$ agrees with a polynomial in k of degree d, called the *Ehrhart polynomial* of P. Thus,

$$i_P(k) = c_d k^d + c_{d-1} k^{d-1} + \cdots + c_1 k + c_0,$$

Definition

Two polytopes P and Q are $\mathrm{GL}_3(\mathbb{Z})$-equidecomposable if they can each be partitioned into finitely many rational simplices

$$P = \bigsqcup P_i, \qquad Q = \bigsqcup Q_i,$$

such that for each i there exists a unimodular map $U_i \in \mathrm{GL}_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ with

$$U_i(P_i) = Q_i.$$

A lattice k-simplex is unimodular if its edge vectors can be extended to a basis of $\mathbb{Z}^n$ .An i-dimensional simplex $\Delta \subset \mathbb{R}^n$ is called half-unimodular if 2Δ is a unimodular lattice simplex. The Ehrhart polynomial, denoted as $i_P(k)$ (or sometimes $ehr(P; k)$), is constructed by summing the Ehrhart polynomials of component simplices(half-unimodular simplices).

Lemma

Every lattice 3-polytope admits a partition into(relatively open)half-unimodular simplices.

Result

The paper by Erbe–Haase–Santos shows that Ehrhart-equivalent 3-polytopes are $\mathrm{GL}_3(\mathbb{Z})$ -equidecomposable.

Definition

A polytope P in$\mathbb{R}^d$ is the convex hull of a finite number of points$\{A_1, \ldots A_n\}$.The polytope P is called integral (resp. rational) if the A_is can be chosen in $\mathbb{Z}^d$ (resp. in $\mathbb{Q}^d$). Denote by $\mathrm{G}_d = \mathbb{Z}^d \ltimes GL(d, \mathbb{Z})$ the group of affine unimodular maps (affine linear isomorphisms preserving the lattice $\mathbb{Z}^d$).Let P be an integral polytope in $\mathbb{R}^d$. The Ehrhart polynomial i_P is clearly invariant by the group G_d . Let $\mathscr{G}_d$ be the pseudogroup associated to G_d and P and Q two integral polytopes. Define a map φ : $P \to Q$ belongs to $\mathscr{G}_d$ (or "is locally in G_d ") if 1.φ is a homeomorphism 2. there exists a rational triangulation $\mathscr{T}$ of P (resp.$\mathscr{T}'$ of Q) such that on the interior of each simplex σ of top dimension of $\mathscr{T}$, φ coincides with an element of G_d , and $\varphi(\sigma) \in \mathscr{T}'$.

Main Conjecture

In dimension $d \geq 3$, there exist integral polytopes P and Q with the same Ehrhart polynomial, but which are not equivalent under the pseudogroup $\mathscr{G}_d$.

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