Statistics and Simulations in the Kardar-Parisi-Zhang class

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The KPZ Universality Class

The Kardar-Parisi-Zhang universality class includes of a class of growing interfaces with height functions satisfying the following:

1. Growth smooths out over time
2. The growth rate is rotationally invariant and slope dependent
3. Independent and identically distributed noise roughens the interface

Randomly growing interfaces that are governed by slope-dependent growth, smoothing, and white-noise random forcing are conjectured to lie in the KPZ Universality Class, meaning that

$$\frac{h(t, t^{2/3}x) - t^1\rho(x)}{\eta(x)t^{1/3}} \to Distribution(h_0)$$

where the distribution on the right hand side depends *only* on the initial condition h_0. The terms $\rho(x)$ and $\eta(x)$ are model-dependent terms similar to mean and standard deviation. Here are some real world examples where the Kardar-Parisi-Zhang universality class can be applied:

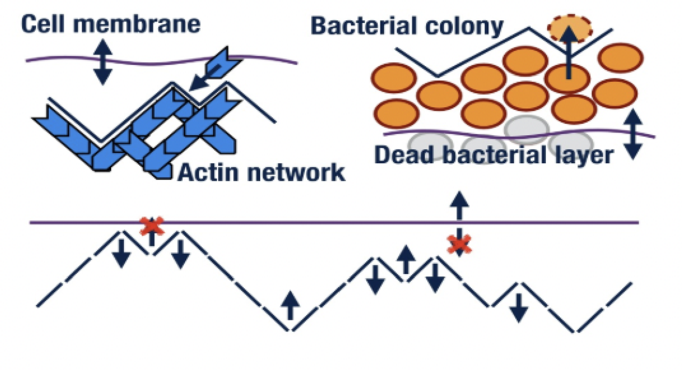

Figure 1. Real-world examples of KPZ behavior include bacterial colony growth

For some choices of initial condition h_0, the limiting distribution comes from the Tracy-Widom family of distributions, which describe the fluctuations of the largest eigenvalues of certain random matrices.

Because of universality, we may get insight into the entire KPZ Universality Class by studying *any* model in the KPZ Universality class. Thus, it makes sense to investigate the simplest models. One such model is the last passage percolation (LPP).

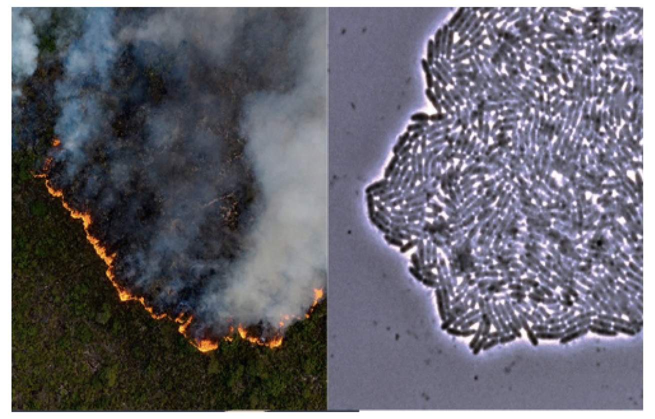

Figure 2. Forest fire growth and traffic flow patters can also be modeled through KPZ behavior.

Last Passage Percolation

Last Passage Percolation (LPP) is a probabilistic model used to study the geometry of optimal paths through random environments. Despite the underlying randomness and microscopic details of the weights assigned to each point or edge, these systems often exhibit universal statistical behavior on a macroscopic scale.

In its classical formulation, LPP is defined on the lattice $\mathbb{Z}^2_{\geq 0}$, where each point (i, j) is assigned an i.i.d. nonnegative random weight $w(i, j)$. A path is constrained to move only rightward or upward. The last passage time from $(0, 0)$ to (m, n) is given by:

$$G(m, n) = \max_{\pi:(0,0)\to(m,n)} \sum_{(i,j)\in\pi} w(i, j).$$

This object $G(m, n)$ represents the maximal total weight collected along a directed up-right path. The path that achieves this maximum is called the geodesic.

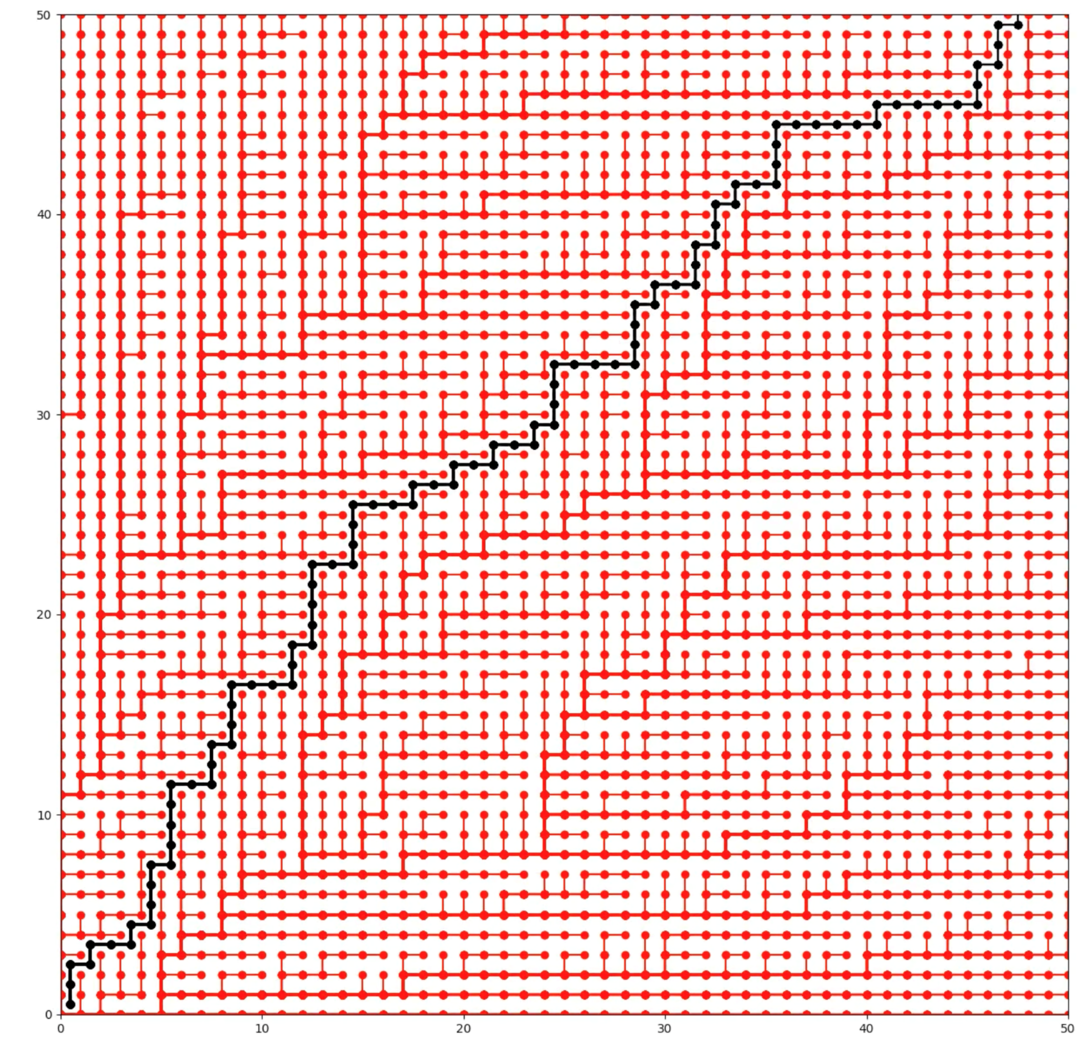

Figure 3. A realization of the LPP model on a 2D grid. The black staircase path is the geodesic from $(0, 0)$ to (m, n) of the competition interface (the boundary that separates different geodesic paths).

Interestingly, LPP is mathematically equivalent to a classical tandem queueing system. Here, each row in the grid corresponds to a service station, and each column corresponds to a customer. The value $w(i, j)$ represents the service time of customer j at station i. Customers proceed through the queues in order, and cannot be served until both their previous service has completed and the station is free. The departure time $D(i, j)$ satisfies:

$$D(i, j) = \max(D(i-1, j), D(i, j-1)) + w(i, j).$$

This recurrence matches the LPP definition of $G(i, j)$, making $G(i, j) = D(i, j)$ a valid identity. Thus, LPP models the departure time of the last customer from the final queue.

Queueing Theory

Queueing theory provides a powerful framework for modeling systems in which entities (often called "customers") arrive over time and require service from a server. These models are governed by two fundamental sequences:

- **Inter-arrival times** (a_i): the time between the arrival of customer $i-1$ and customer i.
- **Service times** (s_i): the time it takes to serve customer i.

From these inputs, one derives important performance quantities:

- **Waiting time** w_i: time spent waiting before service begins.
- **Sojourn time** $J_i = w_i + s_i$: total time in the system.
- **Inter-departure time** D_i: the time between the departures of customers $i-1$ and i.

The system evolves according to simple recursive rules. For example, assuming customer 1 arrives to an empty queue:

$$J_k = s_k + (J_{k-1} - a_k)^+$$

This reflects that a customer only waits if the previous customer hasn't yet departed.

Burke's Theorem plays a central role in equilibrium queueing theory: for an $M/M/1$ queue (exponentially distributed inter-arrival and service times with $\lambda < \mu$), the output inter-departure process is also an i.i.d. exponential process with rate λ—the same as the arrival process.

Multi-class queues generalize this framework by allowing customers from different classes to have different service distributions. Suppose we have n classes, and let I^i be the inter-arrival times for class i, with $I^i \sim \text{Exp}(\rho_i^{-1})$ for $\rho_1 < \rho_2 < \cdots < \rho_n$. Then define:

$$\mathcal{D}(I) = \left(D^{(1)}(I^1), D^{(2)}(I^2, I^1), \ldots, D^{(n)}(I^n, \ldots, I^1)\right)$$

Using an inductive application of Burke's theorem, one can show that the marginal distribution of each $D^{(i)}$ is again $\text{Exp}(\rho_i^{-1})$. This forms a steady-state queueing process with remarkable stability properties. These multi-class dynamics allow researchers to simulate complex queueing systems and understand long-term statistical behavior across classes.

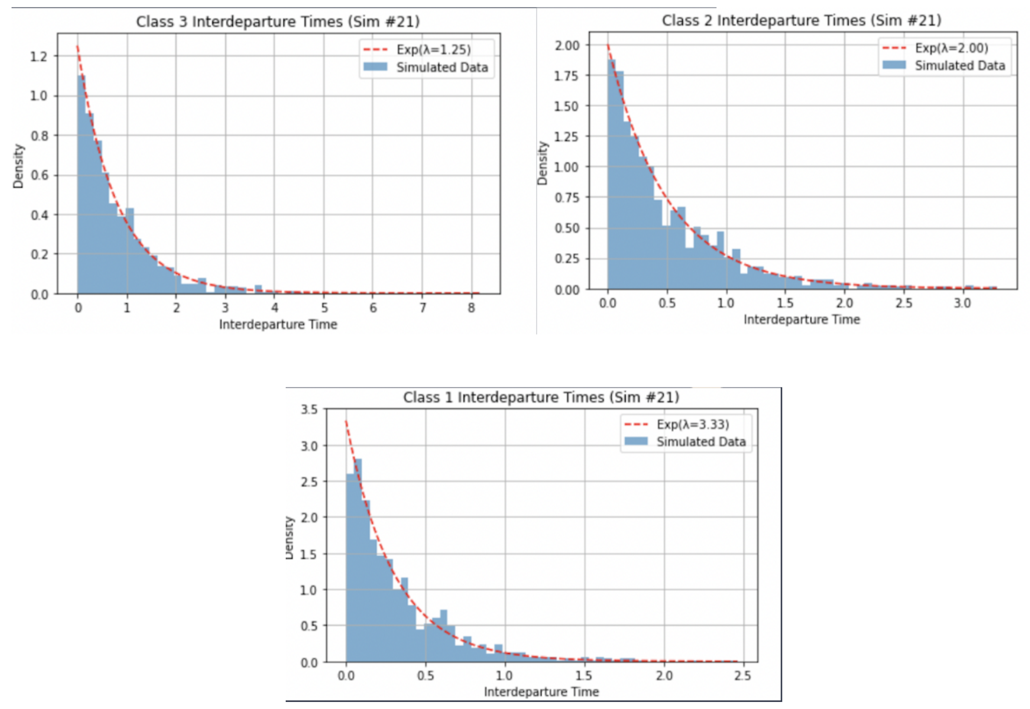

Figure 4. Multi-class Queue of the Departure Times for 3 Priority Classes.

Results

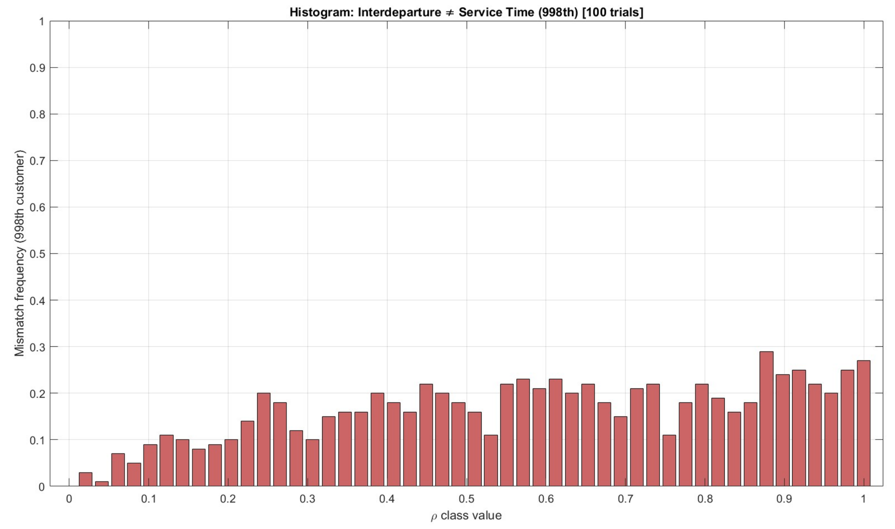

Figure 5. Recorded Frequency of Mismatch Between the Interdeparture Time and Service Time for the 998th Customer.

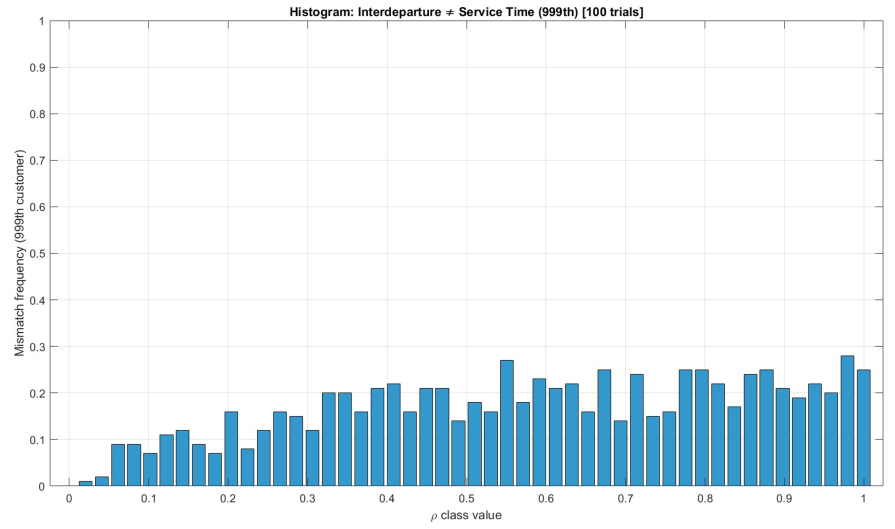

Figure 6. Recorded Frequency of Mismatch Between the Interdeparture Time and Service Time for the 999th Customer..

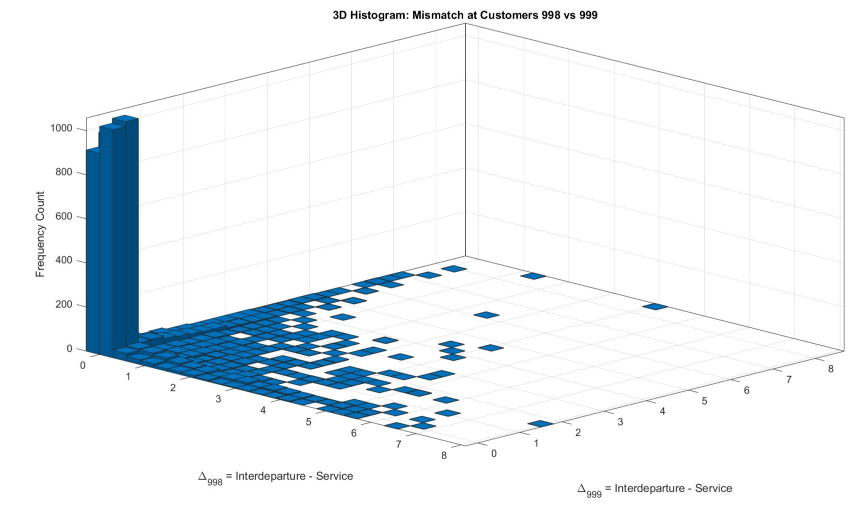

Figure 7. Joint Histogram of the Mismatch Frequency Between the Service and Interdeparture Times of the 998th and 999th Customer.

Future Directions

Moving forward, we aim to study the geometry of this transition more rigorously by analyzing the *competition interface*, a quantity that captures where one class begins to dominate in the service order. In tandem, we will examine the associated *dual service times* to better understand queue dynamics near criticality. This could lead to a deeper probabilistic interpretation of phase transitions in queuing systems and their KPZ connections.