An Explicit Stable Classification for Maximal Tori

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Algebraic Groups on Arbitrary Fields

$\mathrm{GL}_n(\mathbb{R})$, the group of invertible linear transforms on $\mathbb{R}^n$, is the first example of Lie groups. Similar to defining Lie groups as manifolds with group structure, algebraic groups are algebraic varieties (zero sets of polynomials in $\mathbb{A}^n$), where the multiplication and inverse maps are rational polynomial maps. In fact, we can define algebraic groups over any field, including $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}_p$ (p-adic field) and $\mathbb{F}_{p^n}$ (finite fields).

Why Maximal Tori? Motivation and Goal

Maximal tori are maximal Abelian subgroups with regular semisimple elements in algebraic groups over a field F. For example, in $\mathrm{SL}_2(\mathbb{R})$, the different maximal tori are $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. We classify maximal tori up to stable conjugacy, i.e., $P\mathcal{T}_1P^{-1} = \mathcal{T}_2$, where P can have elements in the $\overline{F}$ but $\mathcal{T}_1$, $\mathcal{T}_2$ cannot. In representation theory, much information of an algebraic group is carried in its maximal tori. For $A, B \in \mathrm{GL}_n(F)$, regular semisimple, A and B are stably conjugate iff their characteristic polynomials are the same. In this research, we seek to use the characteristic polynomial to classify the maximal tori an element lies in.

Symplectic groups Sp_{2n} and their Maximal Tori

Under the standard basis of F^{2n}, $\mathrm{Sp}_{2n}(F) \subset \mathrm{SL}_{2n}(F)$ can be defined as

$$\left\{K \in \mathrm{SL}_{2n\times 2n}(F) \colon K^T\Omega K = \Omega\right\}, \text{ where } \Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Note that $\mathrm{Sp}_2(F) = \mathrm{SL}_2(F)$ has 2 types of maximal tori:

- $F^\times$, the multiplicative subgroup of F.
- Kernel of the norm map of an extension $E = F(\sqrt{\theta})$ where the degree of θ over F is 2.

In general, Sp_{2n} has 3 stable conjugacy types of maximal tori:

- $A_1 \times \cdots \times A_k \subset \mathrm{Sp}_{2n_1} \times \cdots \times \mathrm{Sp}_{2n_k}$ where $\sum_{i=1}^k n_i = n$ and each A_i is a maximal torus in Sp_{2n_i}.
- $\left\{\begin{bmatrix} A & 0 \\ 0 & A^{-T} \end{bmatrix} : A \in \mathrm{GL}_n(F),\ p_A \text{ no palindromic factors}\right\}$ for each $E = F(\theta)$, $[E : F] = n$.
- $\left\{\begin{bmatrix} C & \tau D \\ D & C \end{bmatrix} \in M_{2n\times 2n}(F) \colon C, D, \tau \in E \text{ and } C^2 - \tau D^2 = I_n\right\}$ for each $M = E(\sqrt{\tau})$, $[M : E] = 2$, and $E = F(\theta)$, $[E : F] = n$.

Future Work

1. Constructing polynomial conditions to describe Sp_{2n}.
2. Conclude alll maximal tori in G_2 with characteristic polynomials.

Sp_4: Algorithm determining Maximal Tori from polynomials

A matrix $K \in \mathrm{Sp}_4(F)$ has characteristic polynomial $\det(tI_4 - K) = t^4 + \alpha t^3 + \beta t^2 + \alpha t + 1$. While α, β determine the stable conjugacy class of K, we found 3 polynomials indicating the correspondence between α, β and the type of maximal torus K lies in:

$$P_1(\alpha, \beta) = \alpha^2 - 4\beta + 8,$$

$$P_2^{\pm}(\alpha, \beta) = \left(\alpha \pm \sqrt{P_1(\alpha, \beta)}\right)^2 - 16.$$

Given α, β, evaluate P_1:

1. $P_1 \notin (F)^2$: evaluate P_2
  - **1.1** $P_2 \notin (E)^2$: $M|E|F$
  - **1.2** $P_2 \in (E)^2$: GL_2

2. $0 \neq P_1 \in (F)^2$: $\mathrm{SL}_2 \times \mathrm{SL}_2$
  - **2.1** $P_2^i \notin (F)^2$: Π and $E|F$
  - **2.2** $0 \neq P_2^i \in (F)^2$: $F^\times$
  - **2.3** $P_2^i = 0$: $\pm I_2$

3. $P_1 = 0$: $(\mathrm{SL}_2)^2$, evaluate P_2
  - **3.1** $P_2 \notin (F)^2$: $(\Pi)^2$ and $E|F$
  - **3.2** $0 \neq P_2 \in (F)^2$: $(F^\times)^2$
  - **3.3** $P_2 = 0$: $\pm I_4$

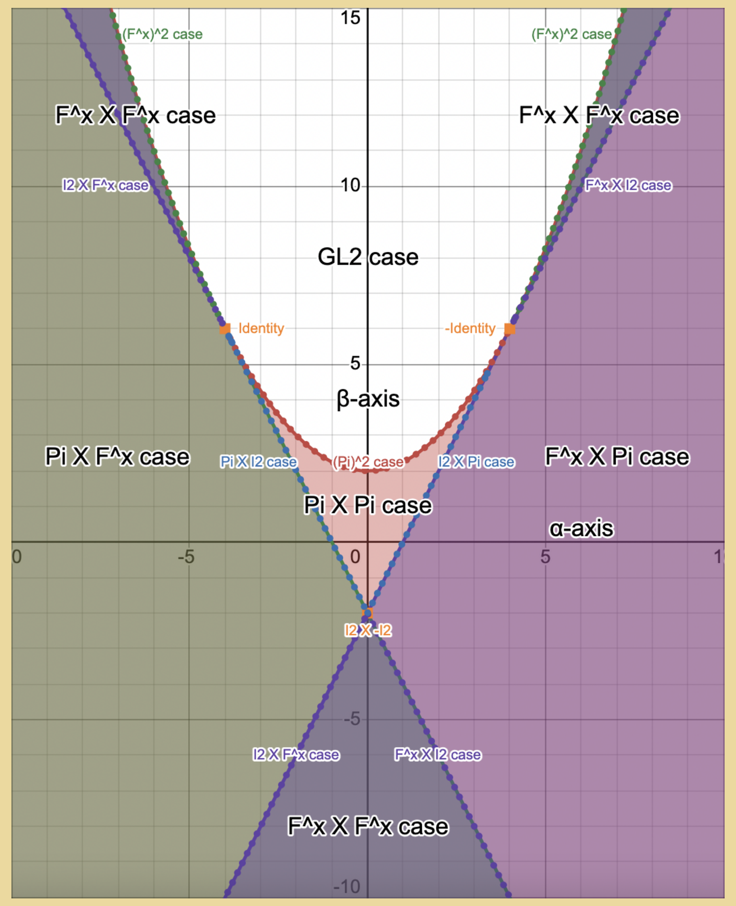

Figure:
Stable Conjugacy for $\mathrm{Sp}_4(\mathbb{R})$

Sp_{2n} Factoring Polynomials

Determining which maximal torus an element is in is surprisingly elementary if we look at how the polynomial factors. If there are any palindromic factors, each factor corresponds to a Sp_{2n_i} block, and if the characteristic polynomial is irreducible, it is in the $M|E|F$ case. If the polynomial has non palindromic factors, the element is in the GL_n case. One can use the substitution $x = t + t^{-1}$ to simplify.

Factoring Palindromic Polynomials over p-adic Fields

By the argument above and the results from Newton polygons, the $M|E|F$ case corresponds to $v_p(\alpha_i) \geq 0$ for all $1 \leq i \leq n$, where α_i are the coefficients of the characteristic polynomial $p_K(t) = t^{2n} + \alpha_1 t^{2n-1} + \cdots + \alpha_1 t + 1$, and v_p is the p-adic valuation. Newton polygons corresponding to the $\mathrm{Sp}_{2n_1} \times \cdots \times \mathrm{Sp}_{2n_k}$ and GL_n case are shown below. Note that the converse doesn't hold in general.

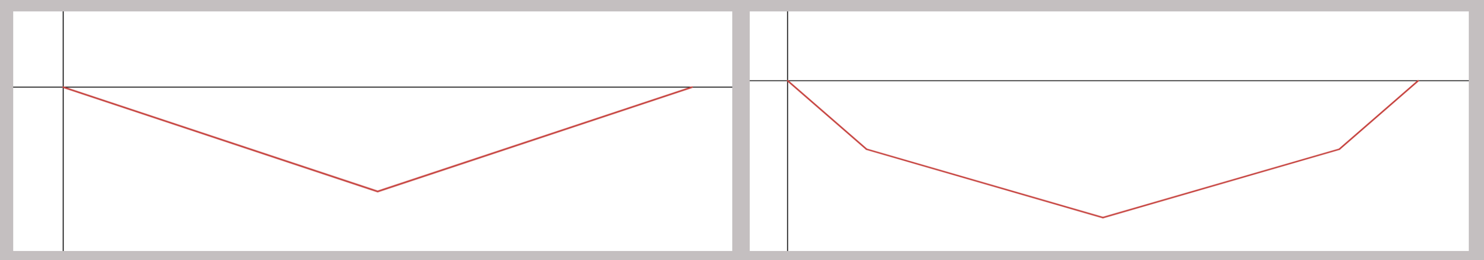

Figure: Newton polygons for GL_n and $\mathrm{Sp}_{2n_1} \times \cdots \times \mathrm{Sp}_{2n_k}$ cases, resp.

The Exceptional Lie Group G_2

In the classification of Lie groups, G_2 is the smallest among the 5 exceptional groups. G_2 is defined as the automorphism group of the Octonion algebra $\mathbb{O}$ and appears in the Langlands programs. On any field F, Zorn's Octonion is a way of constructing an algebra isomorphic to $\mathbb{O}$ based on a 4-tuple with vector and scalar entries, $a, b, c, d \in F$, $\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y} \in F^3$, with the regular dot product and cross product, under pointwise addition and multiplication

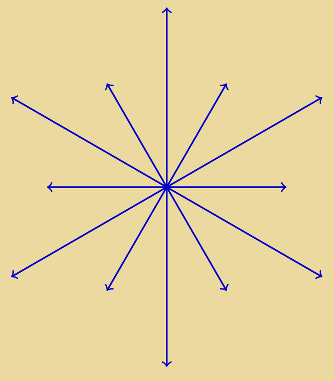

Figure:
G_2 root system

$$\begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix} \cdot \begin{pmatrix} c & \mathbf{x} \\ \mathbf{y} & d \end{pmatrix} = \begin{pmatrix} ac + \mathbf{v}\cdot\mathbf{y} & a\mathbf{x} + d\mathbf{v} - \mathbf{w}\times\mathbf{y} \\ c\mathbf{w} + b\mathbf{y} + \mathbf{v}\times\mathbf{x} & bd + \mathbf{w}\cdot\mathbf{x} \end{pmatrix}.$$

The automorphism group of $\mathbb{O}$ is exactly the split form of G_2 on a base field F. We use the standard basis for Zorn's Octonions as a vector space, $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{e}_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \mathbf{e}_i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$, to represent G_2 as 7×7 invertible matrices satisfying the automorphism conditions.

Maximal Tori in $G_2(F)$ and Their Representation

In split $G_2(F)$, we found maximal subgroups isomorphic to SL_3 and $\mathrm{SL}_2 \times \mathrm{SL}_2/\mathbb{F}_2$. The maximal tori in these two groups are also maximal tori in $G_2(F)$, so we can classify them with the characteristic polynomials of the subgroups. We construct their 7-dim representations based on their action on $\mathbb{O}$.

First, for $g \in \mathrm{SL}_3(F)$ acting on $\mathbb{O}$, the faithful representation in G_2 is $g \mapsto \begin{pmatrix} g & & \\ & g^{-T} & \\ & & 1 \end{pmatrix}$, for $\mathrm{SL}_3(F)$ tori $\begin{pmatrix} a & b\theta & 0 \\ b & a & 0 \\ 0 & 0 & \frac{1}{a^2 - b^2\theta} \end{pmatrix}$, $\begin{pmatrix} a & c\theta & b\theta \\ b & a & c\theta \\ c & b & a \end{pmatrix}$, $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}$.

Secondly, for the maximal subgroup $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)/\mathbb{F}_2$ constructed by two actions which commute with each other and its maximal tori by direct product, we have its embedding into G_2 representation given by

$$\pm\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}, \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\right) \mapsto \begin{pmatrix} x^2 & 0 & 0 & -y^2 & 0 & 0 & -2xy \\ 0 & wx' & -wy' & 0 & -zy' & -zx' & 0 \\ 0 & -wz' & ww' & 0 & zw' & zz' & 0 \\ -z^2 & 0 & 0 & w^2 & 0 & 0 & 2zw \\ 0 & -yz' & yw' & 0 & xw' & xz' & 0 \\ 0 & -yx' & yy' & 0 & xy' & xx' & 0 \\ -xz & 0 & 0 & wy & 0 & 0 & -1+2xw \end{pmatrix}.$$

Given $\mathrm{tr}(g) = \alpha, \mathrm{tr}(g^{-1}) = \beta$, the maximal tori in $\mathrm{SL}_3(F)$ embedded in the 7-dim representation have characteristic polynomials

$$(t-1)(t^3 + \alpha t^2 + \beta t + 1)(t^3 + \beta t^2 + \alpha t + 1).$$

Given $x + w = \gamma, x' + w' = \delta$, maximal tori in $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)/\mathbb{F}_2$ embedded in the 7-dim representation have characteristic polynomials

$$(t-1)\left(t^2 - t\left(\gamma^2 - 2\right) + 1\right)\left(t^4 - t^3\gamma\delta + t^2(\gamma^2 + \delta^2) - t\gamma\delta + 1\right).$$