# Exploring Quantum Graph Invariants

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## Graph Invariants

A graph invariant is some function $f$ such that for graphs $\Gamma_1$ and $\Gamma_2$ ,

$$\Gamma_1 \cong \Gamma_2 \implies f(\Gamma_1) = f(\Gamma_2).$$

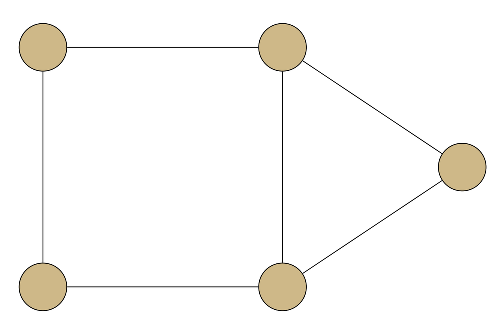


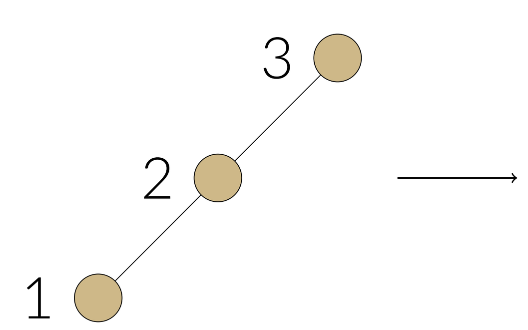
Figure 1. Pictorial representation of two isomorphic graphs

This research focuses on the *Lovász Theta* number, an invariant that bounds the NP-hard to compute chromatic and clique numbers.

$$\alpha(\overline{G}) \leq \vartheta(G) \leq \chi(\overline{G})$$

## Quantum Graphs

To think about a quantum graph, it is helpful to first consider the matricial system associated to a graph:




$$\longrightarrow \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix}$$

Figure 2. Example of matrix system associated to a graph

Generalizing these matricial systems, a *quantum graph* (also called a non-commutative graph) is any matricial system that is closed under taking the adjoint and contains the identity.

## Quantum Graph Invariants

Consider two extensions of *Lovász Theta* for quantum graphs: the CP-index [1] and the quantum Lovász Theta [2].

$$\mathrm{Ind}_{\mathrm{CP}}(\mathcal{S}_1 : \mathcal{S}_2) = \inf\{\|\varphi(I)\| : \varphi(I) \in \mathbb{C}I,\ \varphi(\mathcal{S}_1) \subset \mathcal{S}_2,\ \varphi - I \in \mathrm{CP}(\mathcal{S}_1)\}$$
$$\tilde{\vartheta}(\mathcal{S}) = \max\{\|I+T\| : T \in \mathcal{S}^{\perp}, I+T \succeq 0\}$$

## Research Question

For an arbitrary quantum graph $\mathcal{S}$, does $\mathrm{Ind}_{\mathrm{CP}}(M_n : \mathcal{S}) = \tilde{\vartheta}(\mathcal{S})$?

## Choi Representation

Define $E_{ij}$ to be the matrix whose $i, j$-th entry is 1 and whose other entries are 0. A linear map $\varphi$ is fully defined by the values of $\varphi(E_{ij})$, which can be summarized in a *Choi matrix*:

$$\sum_{i,j} \varphi(E_{ij}) \otimes E_{ij}.$$

**Theorem (Choi):** $\varphi$ is completely positive if and only if its Choi matrix is positive semidefinite.

## Methods

$$\begin{array}{rl} \text{maximize} & \lambda \\ \text{subject to} & \mathrm{tr} \otimes \mathrm{id}(X) = (1-\lambda) I_n \\ & X + \lambda \Delta_n \in M_n \otimes \mathcal{S} \\ & X \in (M_n \otimes M_n)^+ \end{array}$$

$$\downarrow$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0.33 | . | . | 0.17 | −0.17 | −0.17 | 0.17 | −0.17 | −0.17 |
| . | 0.33 | . | −0.17 | 0.17 | −0.17 | −0.17 | 0.17 | −0.17 |
| . | . | 0.33 | −0.17 | −0.17 | 0.17 | −0.17 | −0.17 | 0.17 |
| 0.17 | −0.17 | −0.17 | 0.33 | . | . | 0.17 | −0.17 | −0.17 |
| −0.17 | 0.17 | −0.17 | . | 0.33 | . | −0.17 | 0.17 | −0.17 |
| −0.17 | −0.17 | 0.17 | . | . | 0.33 | −0.17 | −0.17 | 0.17 |
| 0.17 | −0.17 | −0.17 | 0.17 | −0.17 | −0.17 | 0.33 | . | . |
| −0.17 | 0.17 | −0.17 | −0.17 | 0.17 | −0.17 | . | 0.33 | . |
| −0.17 | −0.17 | 0.17 | −0.17 | −0.17 | 0.17 | . | . | 0.33 |

Figure 3. Example Choi Matrix output from the semidefinite program

$$\downarrow$$

$$\varphi(x) = \frac{X \bullet J}{n} I - \frac{X \bullet (J-I)}{n(n-1)} J$$

## Result

Defining $\mathcal{T}_A = (\mathrm{span}\{A\})^{\perp}$, the following are true:

$$\mathrm{Ind}_{\mathrm{CP}}(M_n : \mathcal{T}_{J-I}) = 2,$$
$$\tilde{\vartheta}(\mathcal{T}_{J-I}) = n.$$

Thus the two Quantum Lovász Theta numbers do not agree in general.

## Further Conjectures

$$\mathrm{Ind}_{\mathrm{CP}}(M_n : \mathcal{T}_A) = 2,$$
$$\mathrm{Ind}_{\mathrm{CP}}(M_n : \mathcal{T}_A^*) = n,$$
$$\mathrm{Ind}_{\mathrm{CP}}(\mathcal{T}_A^* : \mathbb{C}I) = n,$$

Given two matrix systems of graphs $\Gamma_1, \Gamma_2$ of size $n \times n$, define the "unitary perturbation" by a unitary matrix $U$ from the Haar distribution by $\Gamma_1 \sigma \Gamma_2; = U \mathcal{S}_{\Gamma_1} U^* + \mathcal{S}_{\Gamma_2}$. Hypothesis:

$$\mathrm{Ind}_{CP}(M_n : \Gamma_1 \sigma \Gamma_2) \leq \min\{\mathrm{Ind}_{\mathrm{CP}}(M_n : \mathcal{S}_{\Gamma_1}),\ \mathrm{Ind}_{\mathrm{CP}}(M_n : \mathcal{S}_{\Gamma_2})\}.$$

One may go a step further in this direction and compute the average inequality over a number of unitary matrices from such Haar distribution as and make the following conjecture:

$$\mathbb{E}[\mathrm{Ind}_{\mathrm{CP}}(M_n : \Gamma_1 \sigma \Gamma_2)] = \mathbb{E}[\tilde{\vartheta}(\Gamma_1 \sigma \Gamma_2)],$$

where the the expected value is over the aforementioned set of unitary matrices.

## Acknowledgments

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## References

[1] R. Araiza, C. Griffin, and T. Sinclair. An index for inclusions of operator systems, 2022.

[2] R. Duan, S. Severini, and A. Winter. Zero-error communication via quantum channels, noncommutative graphs, and a quantum lovász number. *IEEE Transactions on Information Theory*, 59(2):1164–1174, Feb. 2013.