Applications of Singular Value Decomposition

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Singular Value Decomposition

Singular Value Decomposition is a matrix factorization method that takes a complex $m \times n$ matrix, A, and factors it into 3 matrices $U, \Sigma,$ and V.

$$A = U\Sigma V^*$$

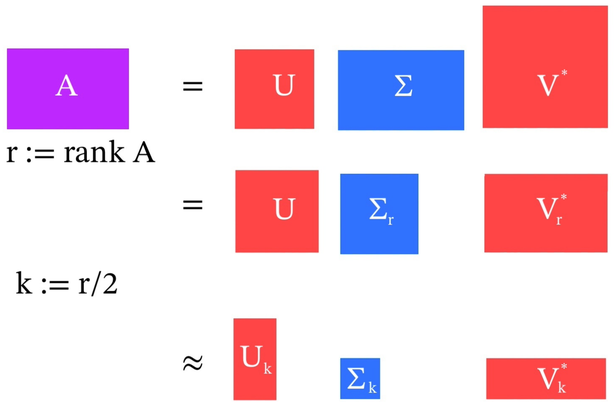
It is found by the diagonalizations of A^*A and AA^*, which are both Hermitian, and more importantly, positive semi-definite (the eigenvalues are non-negative real numbers).
The square roots of the eigenvalues of AA^* and A^*A are called the singular values, σ. We define Σ as an $m \times n$ matrix of zeroes with the singular values along the diagonal sorted in decreasing size. Then the U and V matrices are the unitary eigenvector matrices of AA^* and A^*A respectively. This can be applied to real matrices by replacing each * with ^{T}.

k-rank Approximation With SVD

$$A = U\Sigma V^T = \sigma_1(u_1 v_1^T) + \sigma_2(u_2 v_2^T) + \sigma_3(u_3 v_3^T) + \ldots$$

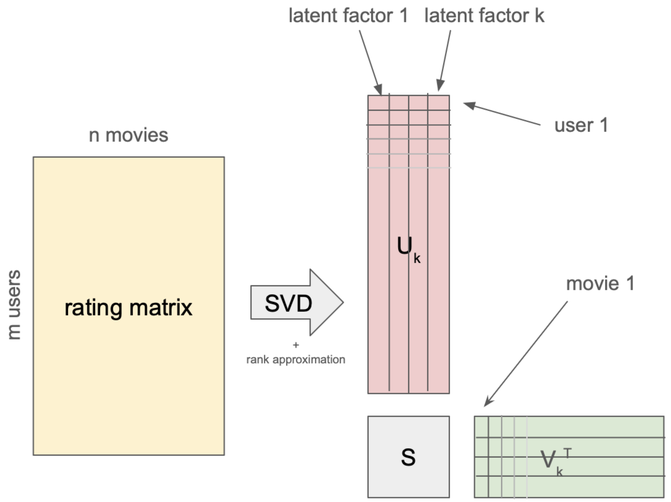
$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq 0$$

Removing all of the $\sigma = 0$ singular values and their associated eigenvectors does not change the decomposition. The smallest nonzero singular values can also be removed without changing the essence of the matrix very much as well. A k-rank approximation of an $m \cdot n$ matrix has a size of $mk + kn$.



Recommender Systems

Missing values in a sparse user-item-rating matrix can be predicted by taking the rank k approximation.



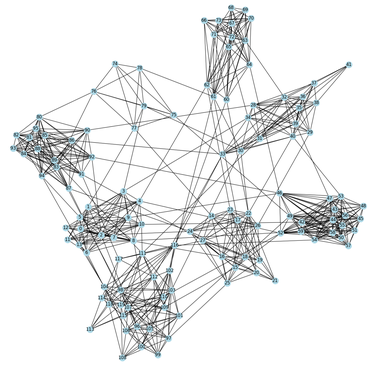
This method to predict the rating for user u and item i refined by Simon Funk to be:

$$\hat{r}_{u,i} = \mu + b_u + b_i + q_u^T p_i$$

The b's represent user and item biases. $q_u^T p_i$ is near equivalent to $U_{row=u}\Sigma V^T_{col=i}$. The new algorithm factorizes matrix A into two matrices PQ^T with k latent factors in each.

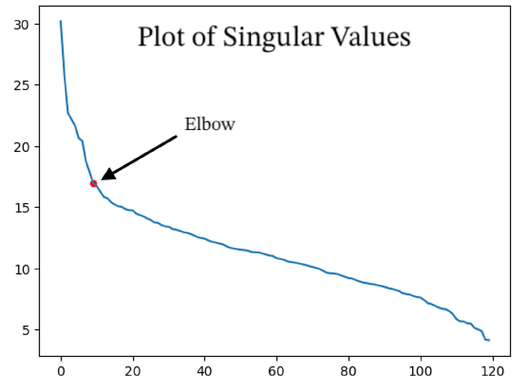
Image Compression

Community Detection of Graphs

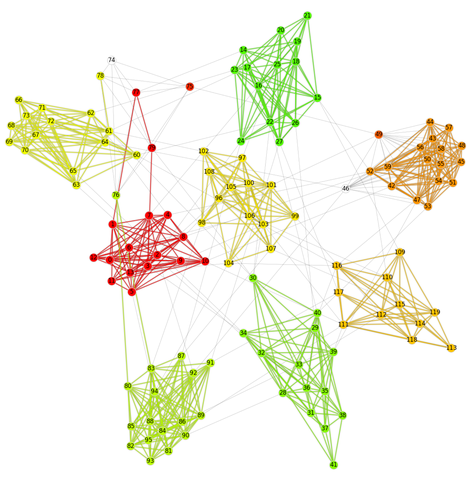


Initial Graph

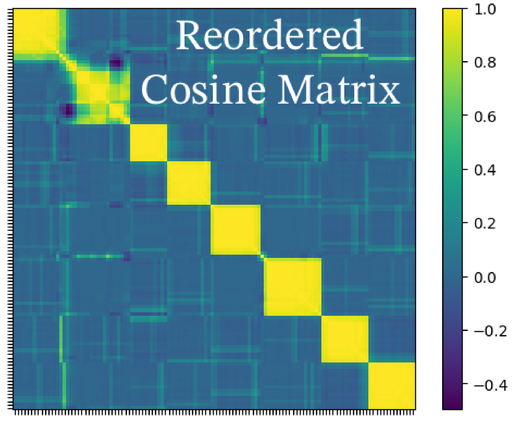
1. Let |L|=D+A; D degree matrix, A adjacency matrix
2. Perform rank k SVD on |L|
3. Build the cosine similarity matrix: $c_{ij} = \frac{v_i \cdot v_j}{|v_i||v_j|}$ for each pair of vectors v_i, v_j in V_k
4. Reorder the cosine matrix



Choose k to be at the "elbow" of the plot



Graph with distinct communities highlighted



Square blocks on the diagonal of the reordered cosine matrix represent candidate communities

References

[1] Martino, L., San Millán-Castillo, R., and Morgado, E. Spectral information criterion for automatic elbow detection. *Expert Systems with Applications 231* (Nov. 2023), 120705.

[2] Sarkar, S., and Dong, A. Community detection in graphs using singular value decomposition. *Physical review. E, Statistical, nonlinear, and soft matter physics 83* (04 2011), 046114.