A Combinatorial Description of the Basilica Julia Set

A Bridge between Combinatorics and Topology

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Abstract: Julia sets are a particular class of fractals that occur by considering complex dynamical systems. The topology of these Julia sets can vary drastically depending on the dynamical system they are associated with. To this end, our goal is to explore the structure of the topology-preserving maps of a given Julia set, given that we know the structure of the associated dynamics.

Dynamics and the Julia set: For a polynomial $f(z)$, the filled Julia set is the set of all points whose forward orbits remain bounded under iteration of f. Its Julia set J_f is defined as the topological boundary of the filled Julia set: $K_f = \{z \in \mathbf{C} : \lim_{n\to\infty} f^n(z) \neq \infty\}$.

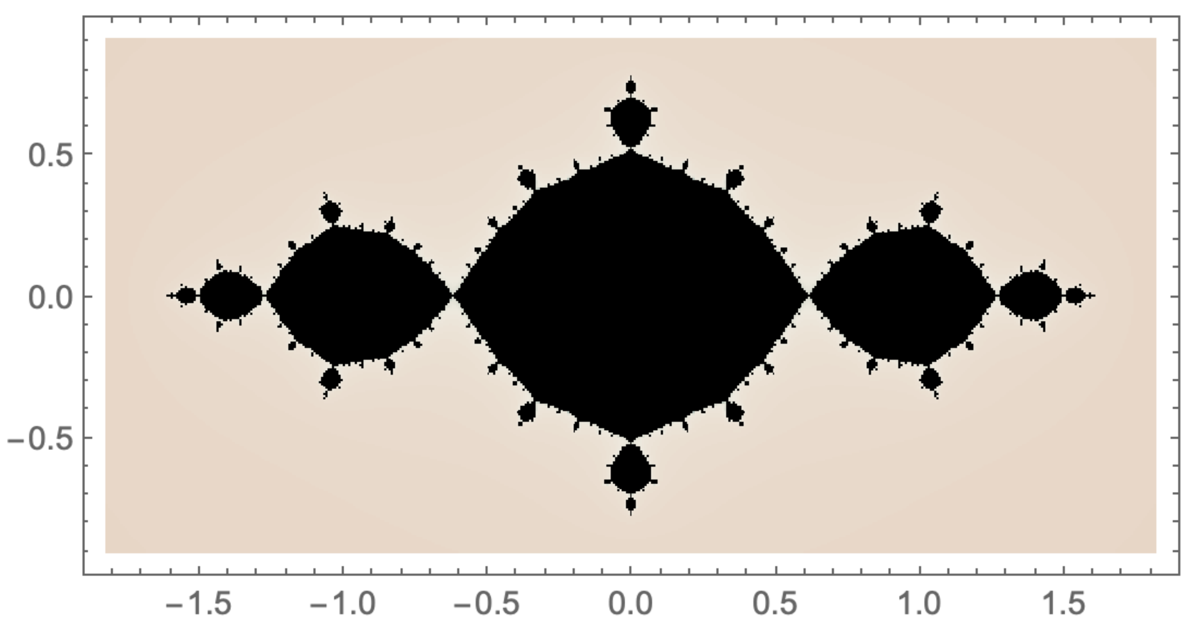


Figure: Filled Julia set for $f(z) = z^2 - 1$ (Basilica)

Dynamical Address: A point is assigned a string of elements of the set $\{T, L, B, R\}$ according to its itinerary in the following partition of the Basilica. Iterating $f(z)$ and recording the itinerary produces its dynamical address. The transition diagram lists all admissible symbolic transitions under the iteration.

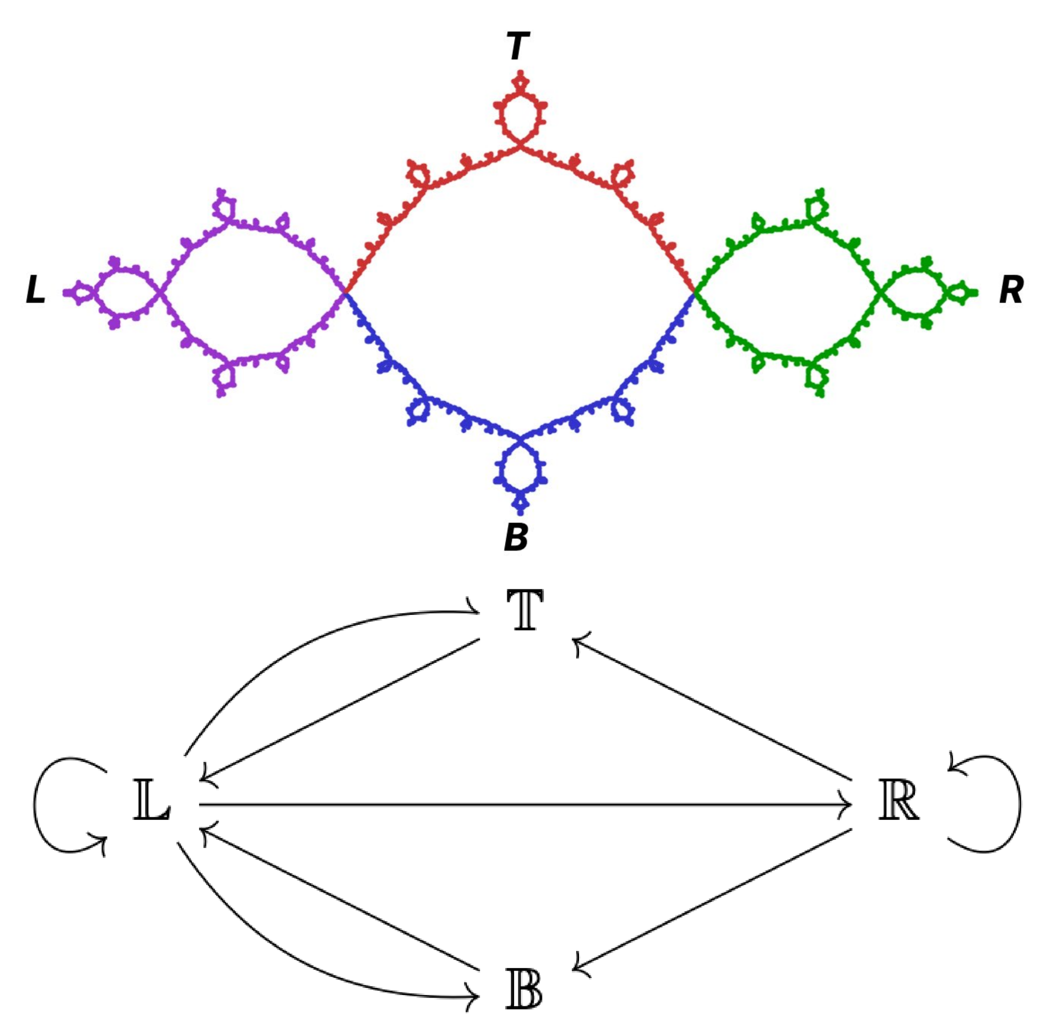
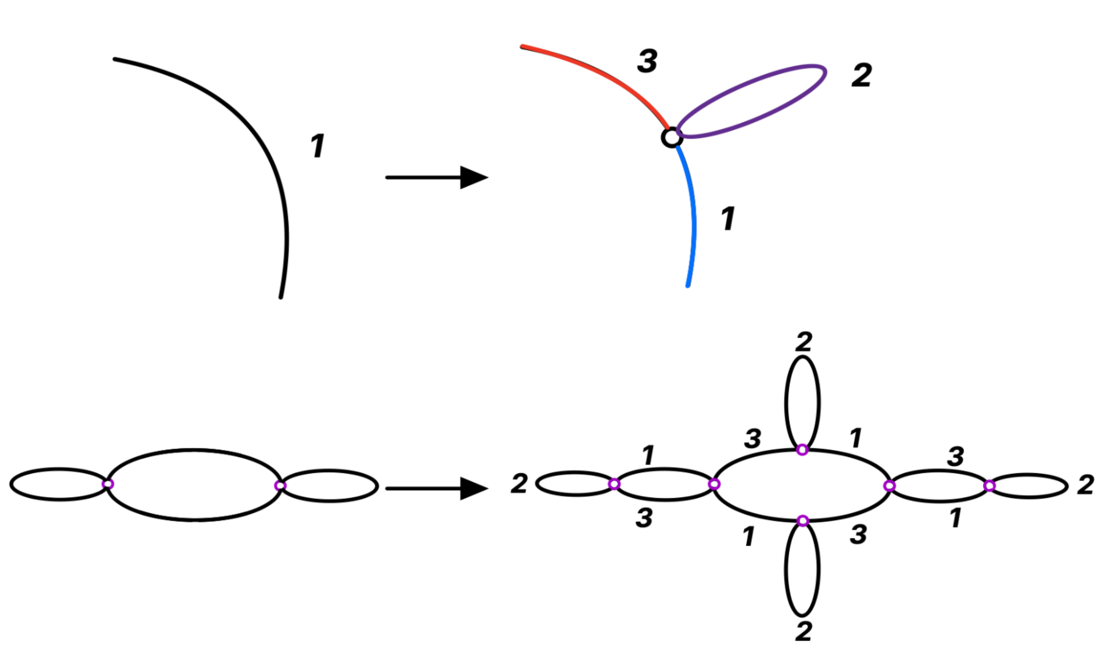


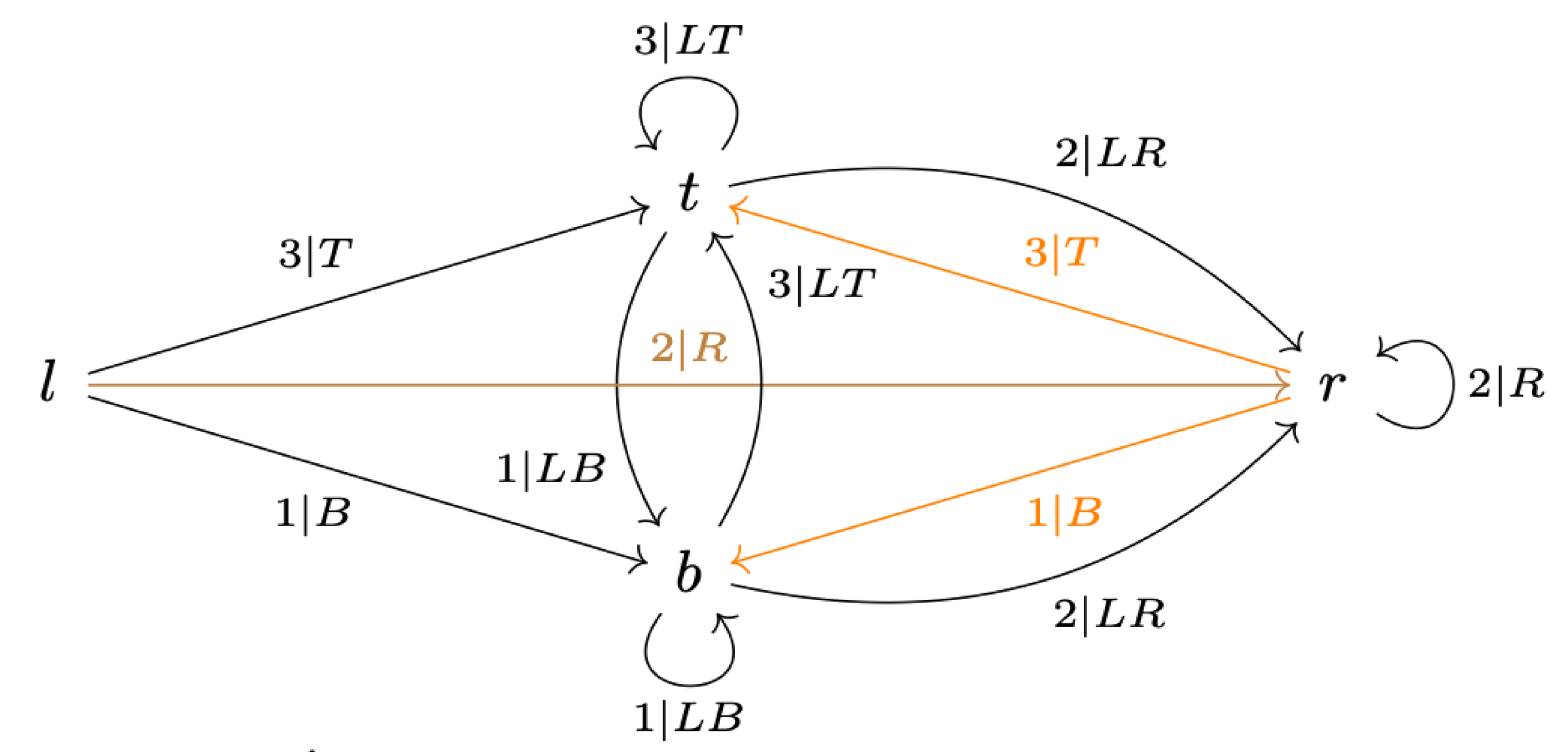
Figure: Dynamical location and Transition Diagram of the Basilica

Geometric Address: Each point is assigned an element of the set $\{t, \ell, b, r\}$ as the first letter by its geometric location (same as dynamical address but in lower case). Then under the replacement rule, each edge is replaced by two edges and a loop, labeled by {1,2,3}. Repeating this process yields the geometric address.



Replacement Rule: sends an edge to a pair of edges connected to a bulb.

A finite state automaton between two addresses



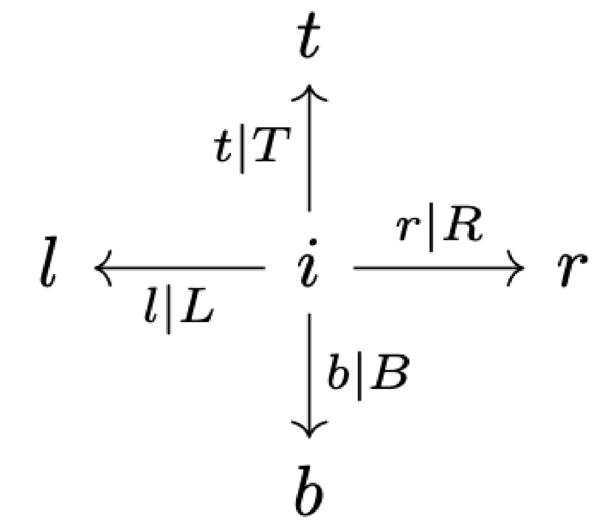


Figure: Above is the geometric address to dynamic address translation between the picture of the Basilica. The left side is the initial states translations.

Theorem: The finite state automata M make the diagram below commute.

Proof (sketch): The bottom part of the diagram is given to us from the properties of quotient maps. The primary goal of this proof is to show the top square commutes.

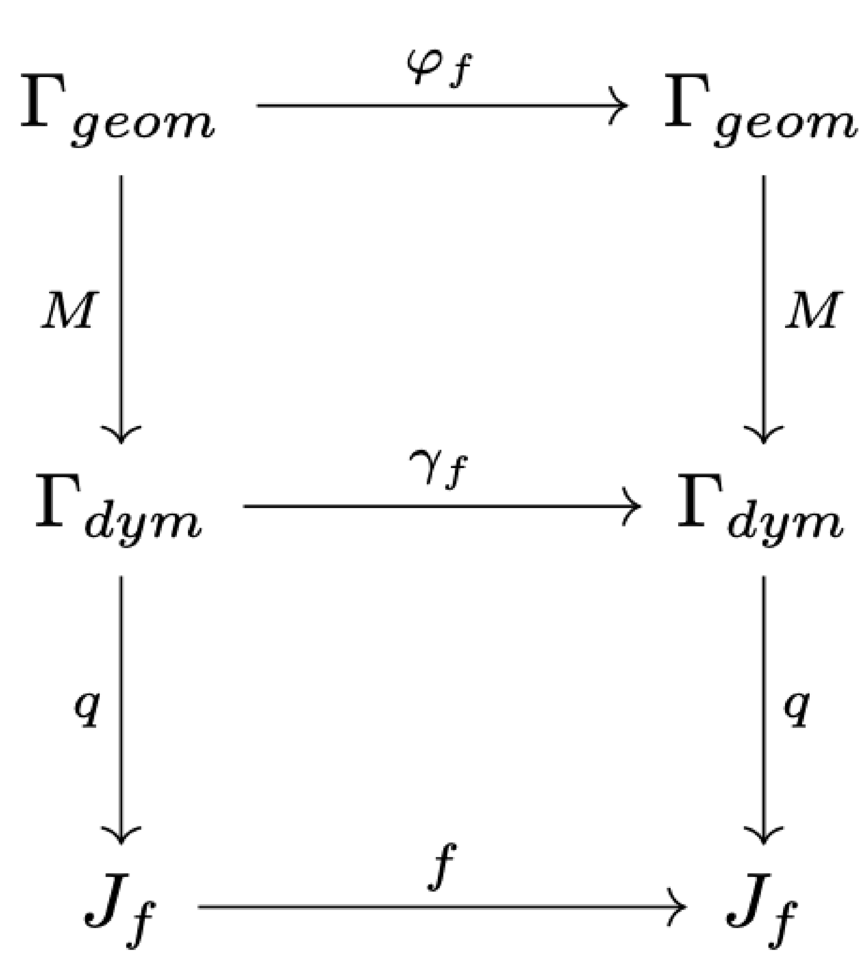
Example Case: Provided a given string in the geometric setting starts with a t, we would like to show that the following equality holds:

$$M \circ \varphi_f(t\omega) = \gamma_f \circ M(t\omega).$$

Key observations for the proof of the $\omega = 1\omega'$ then

- $\varphi_f(t\omega) = \ell\omega$
- M will send $\ell 1$ to LB
- M will send $t1$ to TLB and then under γ_f will send this string to LB

These observations are enough to conclude with our specific choice of $\omega = 1\omega'$, the machine will allow this diagram to commute. The rest of the proof consist of a similar flavor of observations.



$$\gamma_f : \Gamma_{dyn} \to \Gamma_{dyn}$$

$$\gamma_f(a_1 a_2 a_3 a_4 \ldots) = a_2 a_3 a_4 \ldots$$

$$\varphi_f : \Gamma_{geo} \to \Gamma_{geo}$$

$$t\omega \xrightarrow{\varphi_f} l\omega,\ r1\omega \xrightarrow{\varphi_f} b\omega,$$

$$r2\omega \xrightarrow{\varphi_f} r\omega,\ r3\omega \xrightarrow{\varphi} t\omega$$

$$b\omega \xrightarrow{\varphi_f} l\omega,\ l1\omega \xrightarrow{\varphi_f} b\omega,$$

$$l2\omega \xrightarrow{\varphi_f} r\omega,\ l3\omega \xrightarrow{\varphi_f} t\omega$$

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References:

1. Belk, J., Forrest, B. Quasisymmetries of finitely ramified Julia sets. *Math. Ann.* **393**, 1683–1740 (2025).
2. Weinburd, J. Thompson-like groups for quadratic rational Julia sets. Senior Project (2013).