Topological phases and topological field theories

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The math in this talk comes from the study of topological phases of matter in condensed-matter physics.

I’ll first briefly discuss topological phases of matter and what we know about modeling them mathematically.

Lots that we don’t know about modeling topological phases mathematically, but we can extract and solve some questions.
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So condensed-matter theorists set out to classify these phases.
Modeling topological phases

- As usual in condensed-matter physics, use lattice Hamiltonian systems
- Triangulate the ambient manifold $M$
- Use the combinatorial data of the triangulation to write down a Hilbert space $\mathcal{H}$ and Hamiltonian $H : \mathcal{H} \to \mathcal{H}$
- These must be “local,” built out of things which only depend on information within a specified radius
As in QM, $\mathcal{H}$ is the space of states; an eigenvector with eigenvalue $E$ is a state of the system with energy $E$. 

Lowest-energy states, called ground states, are vacuum states, with no particles. The next lowest-energy states correspond to particles localized to specific regions of $\mathcal{M}$. "Next-lowest energy state" requires a gapped Hamiltonian! Two such systems expected to describe same physics ("in the same phase") if one can be deformed into another without closing the gap.
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Difficulties with a straightforward approach

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We’re nowhere near making this a reality

- Usual obstructions to making QFT mathematical
- Also a few new surprises from condensed matter (e.g. fractons)
Low-energy approach

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Low-energy approach

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  - TFT is called the *low-energy limit* of the lattice system
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Example: the toric code

The toric code is the *Drosophila melanogaster* of this field. Let $M$ be a closed $d$-manifold with a triangulation.

- **Fields:** the (discrete) groupoid $\text{Bun}_{\mathbb{Z}/2}(M^1, M^0)$: pairs of a principal $\mathbb{Z}/2$-bundle $P$ on the 1-skeleton of $M$ with a trivialization $\xi$ on the restriction to the 0-skeleton

- **State space** is $\mathcal{H} := \mathbb{C}[\text{Bun}_{\mathbb{Z}/2}(M^1, M^0)]$. We’ll denote states by $\varphi$
Example: the toric code

Given a vertex $v$, let $\psi_v$ be the involution on $\text{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ switching the trivialization of at $v$. Define the operator $A_v : \mathcal{H} \to \mathcal{H}$ by $A_v(\varphi) := \varphi \circ \psi_v$. 

Given a face $f$ and a principal bundle $P \to M^1$, let $\text{Hol}_P(f)$ denote the holonomy of $P$ around $f$. Define the operator $B_f : \mathcal{H} \to \mathcal{H}$ with $B_f(\varphi)(P, \xi) = (-1)^{\text{Hol}_P(f)} \varphi(P, \xi)$. 

The Hamiltonian is $H = \sum_v \frac{1}{2} (1 - A_v) H_v + \sum_f \frac{1}{2} (1 - B_f) H_f$. 
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The Hamiltonian is

\[
H := \sum_v \frac{1}{2} (1-A_v) + \sum_f \frac{1}{2} (1-B_f).
\]
Our ansatz says to get at the toric code using its low-energy TFT

We can see what such a TFT would have to look like by studying the spaces of ground states of the toric code on various manifolds
The ground states of the toric code

- The toric code Hamiltonian is very nice: the $H_v$ and $H_f$ terms are commuting projectors
- This means the space of ground states is the intersection of their kernels
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- This means the space of ground states is the intersection of their kernels.
- $\ker(H_v)$ is functions not depending on the trivialization at $v$.
- $\ker(H_f)$ is the functions which vanish on principal $\mathbb{Z}/2$-bundles $P \to M^1$ which don’t extend across $f$. 
Upshot: the space of ground states is the space of functions on $\pi_0\text{Bun}_{\mathbb{Z}/2}(M)$

This suggests that the low-energy TFT is $\mathbb{Z}/2$ finite gauge theory (aka untwisted $\mathbb{Z}/2$-Dijkgraaf-Witten theory), which assigns to a closed $(n - 1)$-manifold $M$ the space of functions on $\text{Bun}_{\mathbb{Z}/2}(M)$.
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- What we can say mathematically: the state space of this TFT agrees with the space of ground states of the toric code
  - Can’t see the values on (most) bordisms yet
The GDS model

- The *GDS model* is a closely related example
- The $H_\nu$ term is modified by a sign
- This messes up the commutation relations, so the proof we just saw doesn’t work
In this theorem, I will say what the low-energy TFT of the GDS model is (well, to the extent that can be done mathematically) using terms I haven’t defined. I’ll define those terms in the next part of the talk.
Main theorem

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Theorem (D., 2018)

Let $Z : \text{Bord}_n \to \text{Vect}_\mathbb{C}$ be the $\mathbb{Z}/2$-gauge-gravity theory with Lagrangian $\beta$ equal to the degree $n$ part of $w\alpha/(1 + \alpha)$, where $w$ is the total Stiefel-Whitney class and $\alpha \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ is the nontrivial element (thought of as a characteristic class of principal $\mathbb{Z}/2$-bundles).

Then, $Z$ “is” the low-energy TFT of the GDS model, in that on any closed $(n - 1)$-manifold $M$, $Z(M)$ is isomorphic to the space of ground states of the GDS model on $M$, and this isomorphism intertwines the natural $\text{MCG}(M)$-actions on each space.
The cobordism hypothesis says: classifying all TFTs is hard!
Focus on an easier, but still interesting, subclass

**Definition (Freed-Moore)**
A topological field theory $Z: \text{Bord}_n \to \text{Vect}$ is *invertible* if there is another TFT $Z': \text{Bord}_n \to \text{Vect}$ such that $Z \otimes Z' \simeq 1$.

Isomorphism classes of invertible TFTs (IFTs) form an abelian group under tensor product.
Invertible topological phases

- Want to define invertible topological phases (aka *symmetry-protected topological* (SPT) phases) similarly: a phase is invertible if there is another phase such that when you tensor them together, you get the trivial phase.

- “Tensor” is *stacking*, placing both phases on the same material, but with no interactions between them.
  - Explicitly: tensor Hilbert spaces of states together; Hamiltonian is $H := H_1 \otimes 1 + 1 \otimes H_2$.

- Makes sense from physics POV, but not yet a mathematical definition.

- The ansatz specializes: taking the low-energy TFT should produce an equivalence between the classifications of SPT phases and of IFTs.
Classification of invertible TFTs

If $A, B$ are commutative monoids, $f : A \to B$ an invertible homomorphism (i.e. $\text{Im}(f) \subset B^\times$), we can extend $f$ to $K_0(A)$, the abelian group obtained by formally inverting all elements of $A$

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- Maps of abelian groups $K_0(A) \to B^\times$ are in natural bijection with invertible maps $A \to B$
This argument categorifies: replace commutative monoids with symmetric monoidal categories, and replace abelian groups with spectra
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Theorem (Freed-Hopkins, following Galatius-Madsen-Tillmann-Weiss, Schommer-Pries)

There’s a natural isomorphism between the abelian group of $n$-dimensional invertible TFTs with $G$-structure valued in $s\text{Vect}_C$ and $[\Sigma^nMTG_n, \Sigma^n I\mathbb{C}^\times]$. 
Classification of invertible TFTs

- $MTG_n$ is a Madsen-Tillmann spectrum (pull back the negative of the tautological bundle along $BG_n \to BO_n$, take Thom spectrum)

- $I\mathbb{C}^\times$ is the Pontrjagin dual of the sphere, characterized by $[E, \Sigma^n I\mathbb{C}^\times] \cong \text{Hom}_{\text{Ab}}(\pi_n E, \mathbb{C}^\times]$. Note: Freed-Hopkins also prove a version for extended TFTs
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\[ \pi_n(MTG_n) \] is a bordism group, but under a stricter equivalence relation than ordinary \( G \)-bordism.

Thus ordinary bordism invariants define these kinds of bordism invariants, giving IFTs.

Freed-Hopkins prove that for \emph{reflection-positive} IFTs, you get precisely ordinary bordism invariants.

(Modulo a very believable conjecture)
Fix a finite group $G$ and $\beta \in H^n(BG; \mathbb{R}/\mathbb{Z})$. This defines a $\mathbb{C}^\times$-valued bordism invariant of oriented manifolds with a principal $G$-bundle: use the classifying map $f: M \to BG$ to pull back $\beta$; evaluate on the fundamental class, then exponentiate: $\exp(2\pi i \langle f^* \beta, [M] \rangle) \in \mathbb{C}^\times$. These give invertible TFTs called classical Dijkgraaf-Witten theories.

First constructed by Freed-Quinn by other means.

“Classical” here means that the cohomology class plays the role of a Lagrangian in a classical gauge theory.
Some examples

- Slight variant: take $\beta \in H^n(BG; \mathbb{Z}/2)$, multiply it with Stiefel-Whitney classes of $M$, then evaluate and exponentiate as before, defining invertible TFTs called *classical gauge-gravity theories*
  - “gauge-gravity” indicates the Lagrangian has terms corresponding to the principal bundle (“gauge”) and a characteristic class of the underlying manifold (“gravity”)

- Other interesting IFTs from bordism invariants: Arf theory, Arf-Brown-Kervaire theory,…
Some upshots

- Compare this classification of IFTs to preexisting classifications of SPTs by other (physics) methods.
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- Compare this classification of IFTs to preexisting classifications of SPTs by other (physics) methods.
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  - The classifications match, a good sign for the ansatz
- Use this classification to construct invertible TFTs
  - Then use those to construct more TFTs
  - Convenient way to define the TFT I used to get the spaces of ground states of the GDS model
Producing the quantum theory

- To obtain the gauge-gravity theory in the theorem statement, one “quantizes” the classical theory $Z^c_\beta$

- Specifically, a finite form of path integral quantization: sum over principal $\mathbb{Z}/2$-bundles
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- $M$ closed, codimension 0, this is a weighted sum of $Z^{cl}_{\beta}(M, P)$ for $P \in \pi_0 \text{Bun}_{\mathbb{Z}/2}(M)$
- $N$ closed, codimension 1, this is the sections of a vector bundle over the groupoid $\text{Bun}_{\mathbb{Z}/2}(N)$; the fiber at $P$ is $Z^{cl}_{\beta}(N, P)$
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$\mathbb{Z}/2$ is finite, so these are finite sums, hence can be (and are) defined as mathematical operations on TFTs
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2. Comparing with conclusions of Fidkowski, Haah, Hastings, and Tantivasadakarn
3. Studying less-understood variants of topological phases
   - Fractons, higher-order SPTs
   - *Crystalline phases*: the symmetry group can act on space
   - Work in progress comparing a proposal by Freed-Hopkins to physicists’ calculations by other methods