CONSTRUCTING THE VIRASORO GROUPS USING DIFFERENTIAL COHOMOLOGY

ARUN DEBRAY, YU LEON LIU, AND CHRISTOPH WEIS

Abstract. The Virasoro groups are a family of central extensions of $\mathrm{Diff}^+(S^1)$, the group of orientation-preserving diffeomorphisms of S^1, by the circle group $\mathbb{T}$. We give a novel, geometric construction of these central extensions using "off-diagonal" differential lifts of the first Pontryagin class, thus affirmatively answering a question of Freed-Hopkins.

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0. Introduction

Two-dimensional conformal field theories (CFTs) are a Goldilocks zone in mathematical physics: high-dimensional enough to admit many examples and a rich structure, but still mathematically tractable. This is especially true for their groups of symmetries. In all other dimensions, the group of conformal symmetries is finite-dimensional, but in two dimensions, there is an infinite-dimensional family of conformal symmetries: the group $\Gamma := \mathrm{Diff}^+(S^1)$ of orientation-preserving diffeomorphisms of a circle. $\mathrm{Diff}^+(S^1)$ acts on the Hilbert space $\mathcal{H}$ of states that a 2d CFT assigns to a circle, and many important representations of $\mathrm{Diff}^+(S^1)$ arise in this way. However, as is generally the case in quantum physics, these are merely projective symmetries: if $|\psi\rangle \in \mathcal{H}$ and $\mu \in \mathbb{T}$, where $\mathbb{T}$ denotes the unit complex numbers, then the states $|\psi\rangle$ and $\mu|\psi\rangle$ are physically equivalent. Therefore the group that actually acts, which is called a *Virasoro group*, is a central extension of $\mathrm{Diff}^+(S^1)$ by $\mathbb{T}$. There is an $\mathbb{R}$ worth of Virasoro extensions; which central extension we obtain depends on the CFT we began with.

The Virasoro group extensions $\widetilde{\Gamma}_\lambda$, $\lambda \in \mathbb{R}$, of $\mathrm{Diff}^+(S^1)$ are defined as follows: as spaces, $\widetilde{\Gamma}_\lambda \cong \mathbb{T} \times \Gamma$. However, the multiplication is twisted: as a map $(\mathbb{T} \times \Gamma) \times (\mathbb{T} \times \Gamma) \to \mathbb{T} \times \Gamma$, multiplication obeys the formula

$$(z_1, \gamma_1), (z_2, \gamma_2) \longmapsto (z_1 \cdot z_2 \cdot B_\lambda(\gamma_1, \gamma_2), \gamma_1 \circ \gamma_2), \tag{0.1}$$

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where $B\colon \Gamma\times\Gamma\to\mathbb{T}$ is the *Bott-Thurston cocycle* [Bot77]

$$B_\lambda(\gamma_1,\gamma_2) \coloneqq \exp\left(-\frac{i\lambda}{48\pi}\int_{S^1}\log(\gamma_1'\circ\gamma_2)\,\mathrm{d}(\log(\gamma_2'))\right). \tag{0.2}$$

See §1 for the details, and Remark 1.10 for the relationship to the Virasoro algebras, which may be more familiar.

A similar story can happen for other groups. For example, there is a class of 2d CFTs called *Wess-Zumino-Witten (WZW) models*, given by choosing a compact, simply connected Lie group G and an element $h\in H^4(BG;\mathbb{Z})$,[1] which admit a projective symmetry for the (unbased) loop group LG of G. The corresponding central extensions by $\mathbb{T}$, called *Kac-Moody groups*, are common objects of study in representation theory.[2]

Brylinski-McLaughlin [BM94, §5] give a geometric construction of the Kac-Moody central extensions of loop groups using differential cohomology, and the goal of this paper is to do a similar construction to obtain the Virasoro central extensions of $\mathrm{Diff}^+(S^1)$.[3] First we briefly sketch Brylinski-McLaughlin's construction. Let $\mathbb{Z}(n)$ denote the nth *Deligne complex* [Del71, §2.2], a complex of sheaves on the site Man of smooth manifolds given by

$$\mathbb{Z}(n) \coloneqq \Big(\,0\longrightarrow\mathbb{Z}\longrightarrow\Omega^0\longrightarrow\Omega^1\longrightarrow\cdots\longrightarrow\Omega^{n-1}\longrightarrow 0\,\Big). \tag{0.3}$$

For M a smooth manifold, $H^n(M;\mathbb{Z}(n))$ is naturally isomorphic to the group $\check{H}^n(M)$ of degree-n Cheeger-Simons differential characters.[4] We are interested in $H^k(M;\mathbb{Z}(n))$ when k is not necessarily equal to n. For H a Lie group, possibly infinite-dimensional, let $B_\bullet H$ denote the classifying stack for principal H-bundles (see Example 2.23). Then the group of equivalence classes of central extensions of H by $\mathbb{T}$ can be naturally and explicitly identified with $H^3(B_\bullet H;\mathbb{Z}(1))$ [ADH21, Corollary 17.3.3] (here Lemma 2.31). In the case at hand, $H=LG$ is the loop group of G, where G is a compact, connected finite-dimensional Lie group, and there is an equivalence of stacks $LB_\bullet G\simeq B_\bullet LG$. We will see in Corollary 3.21 that for compact G, the truncation map $t\colon\mathbb{Z}(n)\to\mathbb{Z}$ induces an isomorphism

$$t\colon H^{2n}(B_\bullet G;\mathbb{Z}(n))\longrightarrow H^{2n}(B_\bullet G;\mathbb{Z}) = H^{2n}(BG;\mathbb{Z}), \tag{0.4}$$

where BG is the classifying space of G in the usual sense. Thus the level $h\in H^4(BG;\mathbb{Z})$ used to define the WZW model refines to a class $\widetilde{h}\in H^4(B_\bullet G;\mathbb{Z}(2))$. Using the diagram

$$\begin{tikzcd} S^1\times LB_\bullet G \arrow[r, "q"] \arrow[d, "p"] & B_\bullet G \\ LB_\bullet G, & \end{tikzcd} \tag{0.5}$$

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[1] Really, the WZW model is given by a class in $H^3(G;\mathbb{Z})$, but when G is compact and simply connected, the transgression map $H^4(BG;\mathbb{Z})\to H^3(G;\mathbb{Z})$ is an isomorphism. We will find it more useful to use $H^4(BG;\mathbb{Z})$ in this section.

[2] The WZW models still have a projective $\mathrm{Diff}^+(S^1)$-action, and the formulas for the central extensions of $\mathrm{Diff}^+(S^1)$ and of LG that appear for a particular choice of h are related by the *Segal-Sugawara formula*; see [KZ84] for more information.

[3] Unlike the Virasoro central extensions that are the focus of this paper, the Kac-Moody central extensions are topologically nontrivial and do not have a description by global cocycles. These facts lead to difficulties in the Kac-Moody case that are not present for the Virasoro extensions.

[4] Cheeger-Simons' indexing convention differs from ours, and both conventions appear in the literature; we follow [ADH21]. In our convention, the characteristic class map is degree-preserving, with signature $\check{H}^n(M)\to H^n(M;\mathbb{Z})$.

where q is the evaluation map and p is projection onto the second factor, we can pull back along q and then push forward along p. The latter operation is realized by integrating over the S^1 fiber. This defines a *transgression map* for compact, connected G:

$$\tau\colon H^4(B_\bullet G;\mathbb{Z}(2)) \longrightarrow H^4(S^1\times LB_\bullet G;\mathbb{Z}(2)) \longrightarrow H^3(B_\bullet LG;\mathbb{Z}(1)). \tag{0.6}$$

Brylinski-McLaughlin showed that $\tau(\widetilde{h})$ recovers the Kac-Moody central extension of LG at level h, which acts on the WZW theory associated to G and h.

Freed-Hopkins conjectured that a similar procedure could construct the Virasoro groups [ADH21, Question 17.3.8]. Specifically, let $\mathrm{GL}_n^+(\mathbb{R})$ be the group of invertible, orientation-preserving $n\times n$ matrices and begin with the first Pontryagin class $p_1\in H^4(B\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z})$. As $\mathrm{GL}_n^+(\mathbb{R})$ is not compact, lifting to Deligne cohomology is not automatic, but by [ADH21, §17.3] (here Corollary 3.22) there is an affine line of lifts $\hat{p}_1^\lambda\in H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}(2))$, labeled by $\lambda\in\mathbb{R}$. Furthermore, there is a distinguished lift $\hat{p}_1$ that satisfies the (differential) Whitney sum formula (Lemma 3.37).[5]

Let $E\to B_\bullet\mathrm{Diff}^+(S^1)$ be the universal circle bundle and $V\to E$ its vertical tangent bundle; E is the stacky quotient $S^1/\mathrm{Diff}^+(S^1)$. Because the $\mathrm{Diff}^+(S^1)$-action on S^1 is orientation-preserving, V is oriented, so there is a classifying map $q: E\to B_\bullet\mathrm{GL}_1^+(\mathbb{R})$, and we have the following diagram:

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "p"] & B_\bullet\mathrm{GL}_1^+(\mathbb{R}) \\ B_\bullet\mathrm{Diff}^+(S^1). \end{tikzcd} \tag{0.7}$$

Once again, we can pull back and integrate over the fiber to give a map

$$\int_{S^1}\circ\, q^*\colon H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2)) \longrightarrow H^3(B_\bullet\mathrm{Diff}^+(S^1);\mathbb{Z}(1)). \tag{0.8}$$

So after choosing a lift $\hat{p}_1^\lambda$ of p_1, we obtain a central extension of $\mathrm{Diff}^+(S^1)$ given by the class $\int_{S^1}\hat{p}_1^\lambda(V)\in H^3(B_\bullet\mathrm{Diff}^+(S^1);\mathbb{Z}(1))$ — we suppress q^* from notation. Freed-Hopkins' conjecture asserts that the family of extensions of $\mathrm{Diff}^+(S^1)$ given by all choices of $\hat{p}_1^\lambda$ is precisely the family of Virasoro extensions.

Theorem 5.2. *The transgression homomorphism*

$$\begin{gathered} H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2)) \to H^3(B_\bullet\mathrm{Diff}^+(S^1);\mathbb{Z}(1)) \\ \hat{p}_1^\lambda \mapsto \int_{S^1}\hat{p}_1^\lambda(V) \end{gathered} \tag{0.9}$$

maps the $\mathbb{R}$ worth of lifts of p_1 isomorphically to the $\mathbb{R}$ of Virasoro central extensions of $\mathrm{Diff}^+(S^1)$. Furthermore, it takes the distinguished differential lift $\hat{p}_1$ to the Virasoro central extension $\widetilde{\Gamma}_{-12}$ with central charge -12.

Our proof proceeds mostly at the cocycle level. First, though, we prove Lemma 4.4, that the lifts of p_1 are in the image of a map $H^2(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\Omega^1)\to H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2))$. This allows us to compute the transgression map at the level of differential forms. To do so, we model the universal circle bundle E over $B_\bullet\mathrm{Diff}^+(S^1)$ as the stacky double quotient $\mathrm{Diff}^+(S^1)\backslash F/\mathrm{GL}_1^+(\mathbb{R})$, where F is the frame bundle of S^1. This double quotient has a bisimplicial presentation resolving

[5] The case $n=1$ is special: the 1st Pontryagin class of $\mathrm{GL}_1^+(\mathbb{R})$ is trivial and we have a one-dimensional vector space of differential lifts. See Corollary 3.28.

both the left $\mathrm{Diff}^+(S^1)$- and right $\mathrm{GL}_1^+(\mathbb{R})$-actions. It interpolates between the simplicial objects corresponding to the two actions. We chase the generator of the $\mathbb{R}$ worth of differential lifts across this double complex to obtain a form that is easier to integrate over the S^1 fibers. Then we integrate it and see that we obtain the Bott-Thurston cocycle.

Outline. Our first few sections review information that we need to perform the computation. In §1, we introduce the Virasoro groups and the Bott-Thurston cocycles that define them. We collect differential cohomology information in §2, where we introduce the Deligne complexes $\mathbb{Z}(n)$ and prove a few quick lemmas about them. In §3, we study lifts of characteristic classes to $H^{2n}(B_\bullet G;\mathbb{Z}(n))$. The key fact in this section is Theorem 3.12, a theorem of Bott which allows one to compute $H^{2n}(B_\bullet G;\mathbb{Z}(n))$ in terms of the Chern-Weil homomorphism. We use this in Corollary 3.22 to study the affine line of lifts of p_1 to $H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}(2))$, including the distinguished lift $\hat{p}_1$ which satisfies the Whitney sum formula (Lemma 3.37).

In §4 we compute explicit cocycles for the $\mathbb{R}$ worth of off-diagonal lifts of the first Pontryagin class in $H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2))$. Importantly, Lemma 4.4 allows us to work with Ω^1 instead of $\mathbb{Z}(2)$, simplifying the computation. In §5 we prove Theorem 5.2 by a computation through the bisimplicial object discussed above.

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1. Central extensions of $\mathrm{Diff}^+(S^1)$

Let $\Gamma \coloneqq \mathrm{Diff}^+(S^1)$, the group of orientation-preserving diffeomorphisms of the circle. This is a *Fréchet Lie group* [Mil84]: it admits an atlas of charts valued in Fréchet spaces, and group multiplication and inversion are Fréchet maps. The goal of this paper is to construct a particular family of central extensions of Γ called the Virasoro groups; in this section we discuss some basic information about Γ and its central extensions. See Remark 1.10 for the Lie algebra version of this story, which may be more familiar.

We will also need to know about two subgroups of Γ. First, there is an inclusion $\mathrm{SO}_2 \subset \Gamma$ as rotations. We also have $i\colon \mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \Gamma$ as the real fractional linear transformations acting on $\mathbb{RP}^1 = S^1$.

For each $\lambda \in \mathbb{R}$, there is a Fréchet Lie group central extension of Γ by the circle group $\mathbb{T}$ called a *Virasoro group* and denoted $\widetilde{\Gamma}_\lambda$. As spaces, $\widetilde{\Gamma}_\lambda \cong \mathbb{T}\times\Gamma$.[6] However, the multiplication is twisted: as a map $(\mathbb{T}\times\Gamma)\times(\mathbb{T}\times\Gamma)\to\mathbb{T}\times\Gamma$, it obeys the formula

$$(z_1,\gamma_1),(z_2,\gamma_2) \longmapsto (z_1\cdot z_2\cdot B_\lambda(\gamma_1,\gamma_2), \gamma_1\circ\gamma_2), \tag{1.1}$$

where $B_\lambda\colon \Gamma\times\Gamma\to\mathbb{T}$ is the *Bott-Thurston cocycle* [Bot77]:

$$B_\lambda(\gamma_1,\gamma_2) \coloneqq \exp\left(-\frac{i\lambda}{48\pi}\int_{S^1}\log(\gamma_1'\circ\gamma_2)\,\mathrm{d}(\log(\gamma_2'))\right). \tag{1.2}$$

[6] In fact, the inclusion $\mathrm{SO}_2 \hookrightarrow \Gamma$ is a homotopy equivalence, so all principal $\mathbb{T}$-bundles over Γ are trivializable, including those coming from Fréchet Lie group central extensions.

The symbols in (1.2) deserve further explanation. Given a diffeomorphism $\gamma\colon S^1 \to S^1$ in Γ, we can lift it to a map $\tilde\gamma\colon \mathbb{R} \to \mathbb{R}$ along any covering map $\mathbb{R} \to S^1$. The derivative $\tilde\gamma'\colon \mathbb{R} \to \mathbb{R}_+^\times \subset \mathrm{Hom}(\mathbb{R},\mathbb{R})$ descends to a function $\gamma'\colon S^1 \to \mathbb{R}_+^\times$ which is independent of the choices made. (These maps land in $\mathbb{R}_+^\times$ because we are dealing with orientation-preserving diffeomorphisms.) The function $\log\colon \mathbb{R}_+^\times \to \mathbb{R}$ used in (1.2) is the natural logarithm.

Remark 1.3. Bott's original formula for this cocycle was slightly different:

$$B_\lambda(\gamma_1,\gamma_2) \coloneqq \exp\left(-\frac{i\lambda}{48\pi}\int_{S^1}\log((\gamma_1\circ\gamma_2)')\,\mathrm{d}(\log(\gamma_2'))\right). \tag{1.4}$$

This is equal to the cocycle in (1.2) because $\log((\gamma_1\circ\gamma_2)') = \log(\gamma_1'\circ\gamma_2) + \log(\gamma_2')$ and

$$\int_{S^1}\log(\gamma_2')\,\mathrm{d}\log(\gamma_2') = 0 \tag{1.5}$$

because $\log(\gamma_2')\,\mathrm{d}(\log(\gamma_2')) = \frac{1}{2}\mathrm{d}(\log(\gamma_2'))^2$ is exact.

Remark 1.6. As we mentioned in the introduction, Γ acts projectively on two-dimensional conformal field theories, lifting to actual representations of the Virasoro groups. This allows us to choose a favorite normalization: the constant $\frac{1}{48}$ is chosen so that $\widetilde\Gamma_1$ acts on the theory of bosonic periodic scalars (strings). In physics terms, the normalization is set so the theory of bosonic scalars has central charge 1.

Remark 1.7. Let H be a Fréchet Lie group. Then a central extension of H by $\mathbb{R}$ gives rise to a central extension $\widetilde H$ of H by $\mathbb{T}$ via pushout of groups:

$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d, "\exp(2\pi i -)"'] & \widetilde H_{\mathbb{R}} \arrow[d] \\ \mathbb{T} \arrow[r] & \widetilde H. \end{tikzcd} \tag{1.8}$$

For each $\lambda\in\mathbb{R}$, the Virasoro extension $\widetilde\Gamma_\lambda$ arises in this way from an extension of Γ by $\mathbb{R}$ defined using the $\mathbb{R}$-valued cocycle

$$B_{\lambda,\mathbb{R}}(\gamma_1,\gamma_2) \coloneqq -\frac{\lambda}{96\pi^2}\int_{S^1}\log(\gamma_1'\circ\gamma_2)\,\mathrm{d}(\log(\gamma_2')). \tag{1.9}$$

Remark 1.10. Fréchet Lie groups have a notion of Lie algebras, which are Fréchet spaces; the Lie algebra of Γ is the Fréchet space of smooth vector fields on S^1 with its usual bracket. This is the completion of a Lie subalgebra $\mathfrak{w}$ called the *Witt algebra*, which is the Lie algebra of polynomial vector fields on S^1. The Witt algebra is generated by $L_n \coloneqq -ie^{in\theta}\frac{\partial}{\partial\theta}$, $n\in\mathbb{Z}$, with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{1.11}$$

Differentiating a central extension of (Fréchet) Lie groups produces a central extension of (Fréchet) Lie algebras. Applied to the Virasoro extensions $\widetilde\Gamma_\lambda$, we obtain a family of central extensions $\widetilde{\mathfrak{w}}_\lambda$ of the Witt algebra $\mathfrak{w}$ by $\mathbb{R}$, called *Virasoro algebras*. The set of equivalence classes of central extensions of a Lie algebra $\mathfrak{g}$ by $\mathbb{R}$ is given by the Lie algebra cohomology group $H^2(\mathfrak{g};\mathbb{R})$; in [GF68], Gel'fand-Fuks showed that $\mathrm{CExt}_{\mathbb{R}}(\mathfrak{w}) \cong \mathbb{R}$, with $\lambda\in\mathbb{R}$ corresponding to $\widetilde{\mathfrak{w}}_\lambda$.

As vector spaces, $\widetilde{\mathfrak{w}}_\lambda \cong \mathbb{R}\times\mathfrak{w}$, and we can find Lie algebra cocycles $b_\lambda\colon \mathfrak{w}\times\mathfrak{w}\to\mathbb{R}$ such that the Lie bracket $\widetilde{\mathfrak{w}}_\lambda\times\widetilde{\mathfrak{w}}_\lambda\to\widetilde{\mathfrak{w}}_\lambda$ has the formula

$$[(x_1,y_1),(x_2,y_2)] \coloneqq (b_\lambda(y_1,y_2),[y_1,y_2]). \tag{1.12}$$

where $x_1, x_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathfrak{w}$. With our normalization, these cocycles are defined on generators by

$$b_\lambda(L_m, L_n) \coloneqq \frac{\lambda}{12} m^2(m-1)\delta_{m,-n}. \tag{1.13}$$

Denote the central element generating the copy of $\mathbb{R}$ in $\widetilde{\mathfrak{w}}_\lambda$ by c_λ. Its prefactor λ in the commutator is called the *central charge*. The Virasoro algebras with $\lambda \neq 0$ are all isomorphic to each other: explicitly, define $\widetilde{\mathfrak{w}}_1 \xrightarrow{\cong} \widetilde{\mathfrak{w}}_\lambda$ by sending $L_m \mapsto L_m$ and $c_1 \mapsto \lambda c_\lambda$; this is *not* a map of central extensions, as a map of central extensions must be the identity on the central elements. In addition, these isomorphisms do not lift to isomorphisms of the corresponding Virasoro groups. Nonetheless, because of these identifications, $\widetilde{\mathfrak{w}}_1$ is sometimes referred to as *the* Virasoro algebra.

The set of equivalence classes of Fréchet Lie group central extensions of a Fréchet Lie group G by $\mathbb{T}$ is an abelian Lie group $\mathrm{CExt}_{\mathbb{T}}(G)$; in Lemma 2.31 we will explicitly identify it with a sheaf cohomology group. The Virasoro central extensions define a subgroup $\mathcal{V}ir \subset \mathrm{CExt}_{\mathbb{T}}(\Gamma)$ isomorphic to $\mathbb{R}$.

Theorem 1.14 (Segal [Seg81, Corollary 7.5])**.** *$\mathrm{CExt}_{\mathbb{T}}(\Gamma) \cong \mathrm{CExt}_{\mathbb{T}}(\mathrm{PSL}_2(\mathbb{R})) \times \mathcal{V}ir$.*

In particular, the Virasoro extensions are trivial when restricted to $\mathrm{PSL}_2(\mathbb{R})$. The summand $\mathrm{CExt}_{\mathbb{T}}(\mathrm{PSL}_2(\mathbb{R}))$ is isomorphic to $\mathbb{T}$ [Seg81, §7], so

$$\mathrm{CExt}_{\mathbb{T}}(\Gamma) \xrightarrow{\cong} \mathbb{T} \times \mathbb{R}. \tag{1.15}$$

Remark 1.16. If $\mathrm{CExt}_{\mathbb{R}}(\mathfrak{w})$ denotes the vector space of Lie algebra central extensions of $\mathfrak{w}$ by $\mathbb{R}$, then differentiation defines a map $d\colon \mathrm{CExt}_{\mathbb{T}}(\Gamma) \to \mathrm{CExt}_{\mathbb{R}}(\mathfrak{w})$. So a more refined version of Theorem 1.14 is that if $i\colon \mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \Gamma$ is the inclusion map, then

$$(i^*, d)\colon \mathrm{CExt}_{\mathbb{T}}(\Gamma) \longrightarrow \mathrm{CExt}_{\mathbb{T}}(\mathrm{PSL}_2(\mathbb{R})) \times \mathrm{CExt}_{\mathbb{R}}(\mathfrak{w}) \tag{1.17}$$

is an isomorphism. That is, a central extension of Γ is uniquely characterized by its derivative and its restriction to $\mathrm{PSL}_2(\mathbb{R})$. The Virasoro central extensions are all trivializable when restricted to $\mathrm{PSL}_2(\mathbb{R})$, and (when $\lambda \neq 0$) are nontrivial on the level of Lie algebras. Thus the subgroup of Virasoro central extensions is the subgroup $\{0\} \times \mathbb{R}$ under the isomorphism (1.15).

Remark 1.18. The central extensions defining the Virasoro algebras were independently discovered several times. First, Block [Blo66, §2] wrote down a version of (1.13) for a positive-characteristic analogue of $\mathfrak{w}$; then Gel'fand-Fuks [GF68] found cocycles for the Virasoro extensions in characteristic zero. The Virasoro algebra extensions were then rediscovered in physics by Weis (see Brower-Thorn [BT71, §2]).

Given a Lie algebra, it is natural to ask whether it can be exponentiated to a Lie group, and we are not sure who was the first to ask this for the Virasoro algebras. The earliest reference we know of for a cocycle defining the Virasoro group extension is Bott [Bot77].

Remark 1.19. Not everyone means the same thing by "the Virasoro group(s)." Some fix the normalization $\lambda = 1$. Others define the Virasoro groups to be central extensions of Γ by something different. For example, some authors consider central extensions of Γ by $\mathbb{R}$ [Bot77, Lem95, Obl17]; others consider a simply connected version, an extension of the universal cover of Γ by $\mathbb{R}$ [NS15]. Our interest in the Virasoro groups is motivated by the projective Γ-symmetry in 2d conformal field theory, so we do not need to go any farther than $\mathbb{T}$.

2. Background on differential cohomology

The goal of this section is to set up our perspective on differential cohomology. For a more in-depth introduction see [ADH21].

Following Bunke-Nikolaus-Völkl [BNV16], we think of differential cohomology in terms of sheaves of spectra on the site Man of smooth manifolds. In this paper, we only need ordinary differential cohomology, which means we just have to think about sheaves of chain complexes of abelian groups, or equivalently chain complexes of sheaves of abelian groups. Let $\mathsf{Sh}(\mathsf{Man};\mathsf{Ch})$ denote the category of chain complexes of sheaves of abelian groups. If M is a smooth manifold and $E \in \mathsf{Sh}(\mathsf{Man};\mathsf{Ch})$, then the E-cohomology of M, denoted $H^*(M;E)$, refers to the hypercohomology of M with coefficients in E.

Definition 2.1. Let A be an abelian Lie group. We use A to refer to the sheaf of abelian groups on Man whose value on M is the A-valued functions on M, where A carries the discrete topology, and we use $\underline{A}$ to denote the sheaf of A-valued functions where A carries its usual topology.

Ω^k denotes the sheaf of differential k-forms. Note $\underline{\mathbb{R}} \simeq \Omega^0$.

Definition 2.2. The *Deligne complex* $\mathbb{Z}(n)$ [Del71, §2.2] is defined as follows:

$$\mathbb{Z}(n) \coloneqq \Big(0 \longrightarrow \mathbb{Z} \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \cdots \longrightarrow \Omega^{n-1} \longrightarrow 0 \Big) \tag{2.3}$$

where the map $\mathbb{Z} \to \Omega^0$ realizes a $\mathbb{Z}$-valued function as an $\underline{\mathbb{R}}$-valued function, which is the same thing as a 0-form. The map $\Omega^k \to \Omega^{k+1}$ is given by the exterior derivative.

We define $\mathbb{R}(n)$ analogously, with $\mathbb{R}$ (with the discrete topology) replacing $\mathbb{Z}$ in (2.3). In particular, $\mathbb{Z}(0)$ and $\mathbb{R}(0)$ are the sheaves $\mathbb{Z}$ and $\mathbb{R}$, respectively, whose cohomology is ordinary cohomology in $\mathbb{Z}$, resp. $\mathbb{R}$.

Example 2.4. We also use the complex

$$\mathbb{T}(n) \coloneqq \Big(0 \longrightarrow \underline{\mathbb{T}} \xrightarrow{\ \varphi\ } \Omega^1 \longrightarrow \cdots \longrightarrow \Omega^n \longrightarrow 0. \Big) \tag{2.5}$$

where φ is the map $(1/2\pi i)\,\mathrm{d}\log$. Here, $\mathrm{d}\log : \mathbb{T} \to i\Omega^1$ maps a $\mathbb{T}$-valued function $f \in \mathbb{T}(M)$ to the differential form $\mathrm{d}\log(f) \coloneqq \frac{1}{f}\mathrm{d}f \in i\Omega^1(M)$, for all $M \in \mathsf{Man}$.

Lemma 2.6 ([BM94, Remark 3.6])**.** *There is an equivalence* $\mathbb{T}(n)[-1] \simeq \mathbb{Z}(n+1)$.

Proof. The map of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \underline{\mathbb{R}} \arrow[r, "\mathrm{d}"] \arrow[d, "\exp 2\pi i"'] & \Omega^1 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] \arrow[d, equal] & \Omega^n \arrow[r] \arrow[d, equal] & 0 \\ & 0 \arrow[r] & \underline{\mathbb{T}} \arrow[r, "\frac{1}{2\pi i}\mathrm{d}\log"] & \Omega^1 \arrow[r] & \cdots \arrow[r] & \Omega^n \arrow[r] & 0 \end{tikzcd} \tag{2.7}$$

provides an explicit equivalence. Here we have already used the fact that $\Omega^0 \simeq \underline{\mathbb{R}}$ to identify the top row with $\mathbb{Z}(n+1)$. ◻

Remark 2.8. Brylinski [Bry93, Proposition 1.5.7] established a natural isomorphism between $H^n(M;\mathbb{Z}(n))$ and $\check{H}^n(M)$, the group of Cheeger-Simons differential characters of M [CS85].[7] So these "diagonally-graded" groups are the ordinary differential cohomology groups of M. When

[7]As mentioned in Footnote 4, we use a different indexing convention than Cheeger-Simons.

M is a manifold, the "off-diagonal" groups $H^k(M;\mathbb{Z}(n))$, $k > n$, are not very interesting: Brylinski [Bry93, Theorem 1.5.3] shows that when $k > n$, $H^k(M;\mathbb{Z}(n)) \simeq H^k(M;\mathbb{Z})$. For example, take $n = 1$; then then we have a fiber sequence of sheaves

$$\Omega^0 \to \mathbb{Z}(1) \to \mathbb{Z}. \tag{2.9}$$

Taking cohomology, we get a long exact sequence:

$$\cdots \to H^k(M;\Omega^0) \to H^k(M;\mathbb{Z}(1)) \to H^k(M;\mathbb{Z}) \to H^{k+1}(M;\Omega^0) \to \cdots \tag{2.10}$$

Since M has a partition of unity, $H^k(M;\Omega^0) = 0$ for $p \neq 0$. Therefore $H^k(M;\mathbb{Z}(1)) \simeq H^k(M;\mathbb{Z})$ for $k > 1$. Similarly, the fiber of $\mathbb{Z}(n) \to \mathbb{Z}(n-1)$ is $\Omega^n[n]$, so since $H^k(M;\Omega^n) \cong 0$ for $k \neq n$, we deduce that $H^k(M;\mathbb{Z}(n)) \cong H^k(M;\mathbb{Z})$.

For a general stack X, this argument breaks down: we cannot argue using partitions of unity, and the cohomology groups $H^k(X;\Omega^n)$ may be non-zero. An example is given by $H^2(B_\bullet\mathbb{R};\Omega^1) = \mathbb{R}$.[8] This group classifies central extensions of the sheaf of groups $\mathbb{R} = \Omega^0$ by Ω^1. Evaluated on a manifold M, we obtain an extension of groups classified by the 2-cocycle[9]

$$\begin{aligned} z_1\colon \Omega^0(M) \times \Omega^0(M) &\longrightarrow \Omega^1(M) \\ f, g &\longmapsto f\,\mathrm{d}g. \end{aligned} \tag{2.11}$$

Because there are non-trivial classes in $H^k(X;\Omega^n)$ when $n \leq k$ and X is a stack, it is possible that $H^k(X;\mathbb{Z}(n))$ is not necessarily isomorphic to $H^k(X;\mathbb{Z})$; we will see explicit examples of this in §3.

Proposition 2.12 (Bunke-Nikolaus-Völkl [BNV16, §4.1], see also Hopkins-Singer [HS05, §3.2])**.** *Choose $0 \leq m < n$ and let $t\colon \mathbb{Z}(n) \to \mathbb{Z}(m)$ and $t\colon \mathbb{R}(n) \to \mathbb{R}(m)$ denote the truncation maps, which send Ω^{i-1} in degree i to 0 for $i > m$ and do not change the terms in degrees $i \leq m$. Then the commutative square*

$$\begin{tikzcd} \mathbb{Z}(n) \arrow[r, "t"] \arrow[d] & \mathbb{Z}(m) \arrow[d] \\ \mathbb{R}(n) \arrow[r, "t"] & \mathbb{R}(m) \end{tikzcd} \tag{2.13}$$

is homotopy Cartesian.

Letting $m = 0$, this gives us a square comparing differential cohomology to ordinary cohomology; this is the case we use most often.

Differential cohomology comes with a fiber integration map: if $E \to B$ is a fiber bundle of manifolds with fiber S^1 whose vertical tangent bundle is oriented, fiber integration is a map

$$\int_{S^1} \colon H^k(E;\mathbb{Z}(n)) \longrightarrow H^{k-1}(B;\mathbb{Z}(n-1)). \tag{2.14}$$

[8] We will review the definition of classifying spaces $B_\bullet G$ in Example 2.23.

[9] In §5, we show that the class $[z_1] \in H^2(B_\bullet\mathbb{R};\Omega^1)$ of this cocycle transgresses to the class of the Bott cocycle up to a scalar.

Different constructions of this map have been given in [GT00, DL05, HS05, BKS10, Sch13, BB14, BNV16]. For this paper, we only need to know one thing about this map: consider the diagram

$$\begin{array}{ccc} H^n(E;\Omega^k[-(k+1)]) & \xrightarrow{\varphi} & H^n(E;\mathbb{Z}(k+1)) \\ {\scriptstyle\int_{S^1}}\big\downarrow & & {\scriptstyle\int_{S^1}}\big\downarrow \\ H^{n-1}(B;\Omega^{k-1}[-k]) & \xrightarrow{\varphi} & H^{n-1}(B;\mathbb{Z}(k)) \end{array} \tag{2.15}$$

given by comparing fiber integration with integration of differential forms. The horizontal maps come from a map $\varphi\colon \Omega^k[-(k+1)] \to \mathbb{Z}(k+1)$, which is the fiber of the truncation map $\mathbb{Z}(k+1) \to \mathbb{Z}(k)$.

Lemma 2.16. *The diagram* (2.15) *commutes.*

This is because fiber integration on the part of the Deligne complex coming from differential forms is defined in terms of integration of differential forms (see, e.g., [HS05, §3.5]).

We will also need to work with stacks on Man. We use the term *stack* as a shorthand for ∞-*stack*, a functor of ∞-categories $\mathsf{Man}^{\mathsf{op}} \to \mathsf{sSet}$ satisfying descent with respect to hypercovers. Here Man is the site of smooth (paracompact, finite-dimensional) manifolds with surjective submersions as coverings. As shown in [Lur09, Proposition 6.5.2.14], this ∞-category St is presented by the local injective model structure on simplicial presheaves. By [DHI04, Theorem 6.2], this is equivalent to working with the global injective model structure, and localizing at the collection of all hypercovers. In the remainder of the paper, the localization at hypercovers is left implicit. While crucial for the technical setup of differential cohomology, this will not cloud our computations: the cocycles we work with, as well as the equalities we establish between them live in the 1-category of simplicial presheaves $\mathsf{Man}^{\mathsf{op}} \to \mathsf{sSet}$, and descend via the localization functor.

Remark 2.17. Heuristically, a stack is a functorial assignment of simplicial sets or ∞-groupoids to each manifold that satisfies descent. For example, for any manifold $N \in \mathsf{Man}$, its representable presheaf, which assigns to a test manifold M the discrete set $\mathrm{Map}(M, N)$ of smooth maps from $M \to N$, is a stack. Thus Man is a full subcategory of St as the representable functors.

Fréchet manifolds also define objects in St. Given a Fréchet manifold N, we can define a presheaf $\mathcal{F}(N)$ which takes $N \mapsto \mathrm{Map}_{\text{Fréchet}}(M, N)$, where the set of Fréchet maps $M \to N$ also carries the discrete topology. A theorem of Hain [Hai79] and Losik [Los92, Los94] (see also [Wal12, Theorem A.1.5] and [ADH21, Theorem 3.7.5]) proves that this presheaf $\mathcal{F}(N)$ satisfies descent and that this assignment embeds the category of Fréchet manifolds Fr into St as a full subcategory.

If $X_\bullet$ is a simplicial Fréchet manifold, that is, a functor $X_\bullet\colon \Delta^{\mathsf{op}} \to \mathsf{Fr}$, it defines a simplicial presheaf X^{pre} on Man: the value of this presheaf on a test manifold M is the simplicial set $\mathrm{Map}(M, X_\bullet)$ whose n-simplices are the set $\mathrm{Map}(M, X_n)$, where by Map we mean the set of smooth maps. This need not satisfy descent and form a sheaf. We say that a stack $\mathbf{X}$ *is presented by* the simplicial manifold $X_\bullet$ if $\mathbf{X}$ is the *sheafification* of the simplicial presheaf X^{pre}. We can view $X_\bullet$ as a Čech cover of the stack $\mathbf{X}$. If $E \in \mathsf{Sh}(\mathsf{Man};\mathsf{Ch})$ and $\mathbf{X}$ is a stack presented by $X_\bullet$, then the sheaf cohomology $H^*(\mathbf{X}; E)$ can be computed as the hypercohomology of the triple complex associated to $E^*(X_\bullet)$; this does not depend on the choice of presentation of $\mathbf{X}$. This example is also discussed in [FH13, Example 5.5].

Example 2.18. We are principally interested in group quotients. Let G be a (Fréchet) Lie group and X be a manifold with a smooth right G-action. Then the quotient stack X/G can be presented by the simplicial manifold

$$X \Lleftarrow X \times G \Lleftarrow X \times G \times G \Lleftarrow X \times G \times G \times G \ldots \tag{2.19}$$

The face maps are

$$d_i(x, g_1, \ldots, d_n) = \begin{cases} (x \cdot g_1, g_2, \ldots, g_n) & i = 0 \\ (x, g_1, \ldots, g_i g_{i+1}, \ldots, g_n), & 0 < i < n \\ (x, g_1, \ldots, g_{n-1}), & i = n. \end{cases} \tag{2.20}$$

We will also need to take quotient stacks of left G-actions, producing a mirror-image diagram representing the stacky quotient $G\backslash X$:

$$X \Lleftarrow G \times X \Lleftarrow G \times G \times X \Lleftarrow G \times G \times G \times X \ldots, \tag{2.21}$$

whose face maps are

$$d_i(g_1, \ldots, d_n, x) = \begin{cases} (g_2, \ldots, g_n, x) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, x), & 0 < i < n \\ (g_1, \ldots, g_{n-1}, g_n \cdot x), & i = n. \end{cases} \tag{2.22}$$

As we noted above, it is crucial to sheafify the presheaf defined by the simplicial Fréchet manifold (2.19); this presheaf typically does not satisfy descent, but once sheafified, we obtain a valid stack X/G. This can occur even for $X = *$, as we explain in Example 2.23.

Example 2.23. When $X = *$, the quotient stack $B_\bullet G := */G$ is the classifying stack for principal G-bundles.[10] That is, its value on a test manifold U is the nerve of the groupoid of principal G-bundles on U. Using (2.19), $B_\bullet G$ has the simplicial presentation

$$* \Lleftarrow G \Lleftarrow G \times G \Lleftarrow G \times G \times G \ldots \tag{2.24}$$

We could have just as well made G act on $*$ from the left; we would obtain the same simplicial manifold: the identity map on n-simplices defines an equivalence $G\backslash * \to */G$.

The simplicial presheaf associated to this simplicial presentation classifies trivializable G-bundles and their automorphisms. This presheaf does not satisfy descent, because trivializable G-bundles on an open cover of M can glue to a nontrivial principal G-bundle on M. The sheafification of this simplicial presheaf is $B_\bullet G$, which classifies principal G-bundles and automorphisms: the sheafification replaces the value of $(*/G)^{pre}$ on M with the value of $(*/G)^{pre}$ on a good Čech cover of M. On such a cover, a principal G-bundle is the data of transition functions on double intersections satisfying a cocycle condition on triple intersections, and this is what the presheaf $(*/G)^{pre}$ assigns to this Čech cover. Thus its sheafification $B_\bullet G$ is the classifying stack of principal G-bundles.

Example 2.25. Likewise, let $X = G$ with G-action given by right multiplication. The corresponding quotient stack is known as $E_\bullet G$; it is equivalent to $G/G \simeq *$. $E_\bullet G$ is also the classifying stack of trivial principal G-bundles; that is, its value on a test manifold U is the nerve of the groupoid of trivialized principal G-bundles on U. Forgetting the trivialization defines a map $E_\bullet G \to B_\bullet G$,

[10] $B_\bullet G$ is not the same thing as the constant stack at the simplicial set BG, where we forget the smooth structure and only remember the homotopy type. For example, when G is contractible, $BG \simeq *$ and $H^*(BG; \mathbb{Z}(n))$ is trivial, but we will see in Corollary 3.28 that $H^4(B_\bullet \mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2))$ is nonzero. As another example, the Heisenberg group is an extension of $\mathbb{R}^2$ by $\mathbb{R}$ and defines a nonzero class in $H^2(B_\bullet \mathbb{R}^2, \underline{\mathbb{R}})$, but $H^2(B\mathbb{R}^2; \underline{\mathbb{R}}) = 0$.

and this map is the universal principal G-bundle in the setting of stacks on Man. Using (2.19), $E_\bullet G$ has the simplicial presentation

$$G \leftleftarrows G \times G \leftleftarrows G \times G \times G \leftleftarrows G \times G \times G \times G \dots . \tag{2.26}$$

If A is an abelian group considered with the discrete topology, then $H^*(B_\bullet G; A) \cong H^*(BG; A)$ more or less by definition, and likewise $H^*(E_\bullet G; A) \cong H^*(EG; A) = A$ concentrated in degree 0.

Example 2.27. If G is a finite-dimensional Lie group, there are analogues of $B_\bullet G$ and $E_\bullet G$ for principal G-bundles with connection. $B_\nabla G$ is the stack whose value on a test manifold U is the nerve of the groupoid of principal G-bundles with connection on U, and $E_\nabla G$ is the stack whose value on U is the nerve of the groupoid of trivialized principal G-bundles with connection on U. Forgetting the trivialization defines a map $E_\nabla G \to B_\nabla G$, which is the universal principal G-bundle with connection; in particular, $B_\nabla G$ is the quotient of $E_\nabla G$ by a right G-action. There is an equivalence $E_\nabla G \simeq \Omega^1 \otimes \mathfrak{g}$ [FH13, (5.15)]; under this equivalence, the G-action on $E_\nabla G$ is by gauge transformations:

$$A \cdot g = g^{-1} A g + g^{-1}\, \mathrm{d}g. \tag{2.28}$$

Freed-Hopkins [FH13, Proposition 5.24] show that since $B_\nabla G = E_\nabla G/G \simeq (\Omega^1 \otimes \mathfrak{g})/G$, $B_\nabla G$ is presented by the simplicial object

$$\Omega^1 \otimes \mathfrak{g} \leftleftarrows (\Omega^1 \otimes \mathfrak{g}) \times G \leftleftarrows (\Omega^1 \otimes \mathfrak{g}) \times G \times G \leftleftarrows (\Omega^1 \otimes \mathfrak{g}) \times G \times G \times G \dots \tag{2.29}$$

Because we are interested in extensions of $\mathrm{Diff}^+(S^1)$, we allow G to be an infinite-dimensional Fréchet Lie group in Example 2.23. Brylinski [Bry00, Proposition 1.6] showed that when G is a Fréchet Lie group and A is an abelian Lie group, the group of equivalence classes of central extensions of Fréchet Lie groups

$$0 \longrightarrow A \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 0. \tag{2.30}$$

is naturally isomorphic to the *Segal-Mitchison cohomology group* $H^2_{\mathrm{SM}}(G; A)$ [Seg70, Seg75], and Schreiber [Sch13, Theorem 4.4.36] shows that for $A = \underline{\mathbb{T}}$, $H^2_{\mathrm{SM}}(G; \mathbb{T}) \cong H^2(B_\bullet G; \underline{\mathbb{T}})$.[11]

The Virasoro group is a central extension of $\mathrm{Diff}^+(S^1)$ by $\mathbb{T}$, so we will be interested in $\underline{\mathbb{T}}$-cohomology.

Lemma 2.31. *For G a Fréchet Lie group, equivalence classes of Fréchet Lie group central extensions by $\mathbb{T}$ are naturally identified with $H^3(B_\bullet G; \mathbb{Z}(1))$.*

Proof. Apply Lemma 2.6 for $n = 0$, producing an equivalence $\underline{\mathbb{T}}[-1] \simeq \mathbb{Z}(1)$, hence an equivalence $H^2(B_\bullet G; \underline{\mathbb{T}}) \cong H^3(B_\bullet G; \mathbb{Z}(1))$. □

Remark 2.32. As spaces, Fréchet Lie group central extensions by $\mathbb{T}$ must be principal $\mathbb{T}$-bundles. Forgetting the central extension and just remembering the principal $\mathbb{T}$-bundle over G defines a homomorphism $H^3(B_\bullet G; \mathbb{Z}(1)) \to H^1(G; \underline{\mathbb{T}}) \xrightarrow{\cong} H^2(G; \mathbb{Z})$. This map can also be described as follows: first apply the truncation $\mathbb{Z}(1) \to \mathbb{Z}$ to land in $H^3(BG; \mathbb{Z}) \cong [BG, K(\mathbb{Z}, 3)]$; then take the loop space functor to land in $[\Omega BG, \Omega K(\mathbb{Z}, 3)] \cong [G, K(\mathbb{Z}, 2)] \cong H^2(G; \mathbb{Z})$.

[11] There are several other notions of continuous or smooth cohomology such that H^2 correctly classifies central extensions of topological or Lie groups, including theories due to Segal-Mitchison (*ibid.*), Wigner [Wig73], Moore [Moo76], Flach [Fla08], Fuchssteiner-Wockel [FW12], Khedekar-Rajan [KR12], and Wagemann-Wockel [WW15]. Wagemann-Wockel (*ibid.*, Theorem IV.5) provide a general isomorphism theorem identifying most of these cohomology theories. Contrast this with the notion of globally continuous or smooth cohomology discussed for example in Stasheff [Sta78].

3. Off-diagonal differential characteristic classes

The goal of this section is to study lifts of characteristic classes to the off-diagonal differential cohomology groups $H^{2n}(B_\bullet G;\mathbb{Z}(n))$. We will show that when G is compact, the map $t\colon H^{2n}(B_\bullet G;\mathbb{Z}(n))\to H^{2n}(BG;\mathbb{Z})$ is an isomorphism (Corollary 3.21), and that in general (Corollary 3.19) there is a pullback square

$$\begin{array}{ccc} H^{2n}(B_\bullet G;\mathbb{Z}(n)) & \xrightarrow{t} & H^{2n}(BG;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^*(\mathfrak{g}^\vee)^G & \xrightarrow{j^*} & \mathrm{Sym}^n(\mathfrak{k}^\vee)^K, \end{array} \tag{3.1}$$

where K is a maximal compact in G, $\mathfrak{k}$ is the Lie algebra of K, and j^* is the pullback on functions induced by the inclusion $j:\mathfrak{k}\hookrightarrow\mathfrak{g}$. We then use this to characterize differential lifts of $p_1\in H^4(B\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z})$.

First, though, we review Chern-Weil theory from a perspective that will be convenient when we study off-diagonal characteristic classes. Let G be a Lie group with $\pi_0(G)$ finite, and let $\mathfrak{g}$ be the Lie algebra of G.

Definition 3.2. We let $I^*(G)$ denote the algebra $\mathrm{Sym}^*(\mathfrak{g}^\vee)^G$ of G-invariant polynomials on $\mathfrak{g}$, where G acts by the adjoint action.

Definition 3.3 ([Car50, Che52])**.** Let M be a manifold and $P\to M$ be a principal G-bundle with connection $\Theta\in\Omega^1_P(\mathfrak{g})$; let $\Omega\in\Omega^2_P(\mathfrak{g})$ be the curvature of Θ. The *Chern-Weil map*

$$CW\colon I^*(G)\longrightarrow\Omega^*(M) \tag{3.4}$$

is defined as follows: given $f\in I^k(G)$, we can evaluate f on $\Omega^{\wedge k}\in\Omega^{2k}_P(\mathfrak{g}^{\otimes k})$, giving an element $f(\Omega^{\wedge k})\in\Omega^{2k}_P(\mathbb{R})$; because f is Ad-invariant, $f(\Omega^{\wedge k})$ descends to a form on M, and we define $CW(f)\in\Omega^{2k}(M)$ to be that form.

This map is natural in G and in (P,Θ). Since Ω is a 2-form, this map doubles the grading.

Theorem 3.5 (Chern [Che52], Weil [Wei49])**.** *$CW(f)$ is always a closed form, and its de Rham class does not depend on Θ. CW passes to an algebra homomorphism $CW\colon I^*(G)\to H^*(BG;\mathbb{R})$. When G is compact, this map is an isomorphism.*

The Chern-Weil map refines to on-diagonal differential cohomology:

Theorem 3.6 (Cheeger-Simons [CS85, Theorem 2.2])**.** *For G a compact Lie group and $c\in H^{2n}(BG;\mathbb{Z})$, there is a unique lift of c to a class $\check c\in H^{2n}(B_\nabla G;\mathbb{Z}(2n))$ whose curvature form is the Chern-Weil form associated to the image of c in $\mathbb{R}$-valued cohomology.*

Remark 3.7. Cheeger-Simons did not work with the universal object $B_\nabla G$, but instead with approximations of it. Bunke-Nikolaus-Völkl [BNV16, §5.2] showed Cheeger-Simons' construction lifts to $B_\nabla G$. Freed-Hopkins [FH13] interpret the Chern-Weil map as providing an isomorphism $I^*(G)\xrightarrow{\cong}\Omega^*(B_\nabla G)$.

Beginning with work of Beĭlinson [Beĭ84, §1.7], there is a parallel and different story lifting c to $\tilde c\in H^{2n}(B_\bullet G;\mathbb{Z}(n))$, not requiring connections, and landing in off-diagonal differential cohomology groups.

In this section we will study the truncation map $t\colon H^{2n}(B_\bullet G;\mathbb{Z}(n))\to H^{2n}(B_\bullet G;\mathbb{Z}) = H^{2n}(BG;\mathbb{Z})$. When G is compact, this map is an isomorphism (Corollary 3.21). We are most interested in $G = \mathrm{GL}_n^+(\mathbb{R})$, which is not compact, but we will be able to characterize the preimages of the first Pontryagin class $p_1 \in H^4(B\mathrm{GL}_n^+(\mathbb{R}); \mathbb{Z})$ in Corollary 3.22.

Remark 3.8. Work on characteristic classes in off-diagonal Deligne cohomology began in the setting of algebraic or complex geometry; this includes Beĭlinson's original work [Beĭ84, §1.7] as well as work of Bloch [Blo78], Soulé [Sou89], Brylinski [Bry99a, Bry99b], and Dupont-Hain-Zucker [DHZ00] studying relationships between off-diagonal and on-diagonal differential lifts of Chern classes. In the smooth setting, work began with Bott's calculation of $H^*(B_\bullet G; \Omega^q)$ [Bot73], later reinterpreted in off-diagonal differential cohomology by Waldorf [Wal10] and in [ADH21, Chapters 15–17]. See also work of Shulman [Shu72] and Bott-Shulman-Stasheff [BSS76] studying $H^*(B_\bullet G; \Omega^{\geq q})$.

Lemma 3.9 ([ADH21, §16.1]). *Let G be a finite-dimensional Lie group with $\pi_0(G)$ finite. Then there is a pullback square*

$$\begin{tikzcd} H^{2n}(B_\bullet G; \mathbb{Z}(n)) \arrow[r, "t"] \arrow[d] & H^{2n}(BG; \mathbb{Z}) \arrow[d] \\ H^{2n}(B_\bullet G; \mathbb{R}(n)) \arrow[r, "t"] & H^{2n}(BG; \mathbb{R}), \end{tikzcd} \tag{3.10}$$

where the horizontal maps are induced from the maps in (2.13), where $m = 0$, and we implicitly use the identification $H^*(B_\bullet G; A) \simeq H^*(BG; A)$.

Proof. The pullback square (2.13) with $m = 0$ induces a Mayer-Vietoris sequence for the cohomology of $B_\bullet G$ with coefficients in $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}(n)$, and $\mathbb{R}(n)$. For any finite-dimensional Lie group G with $\pi_0(G)$ finite, $H^{2n-1}(BG; \mathbb{R}) = 0$: retract G onto its maximal compact; then, for compact Lie groups, the Chern-Weil map $I^*(G) \to H^*(BG; \mathbb{R})$ is an isomorphism, and its image vanishes in odd degrees. Since $H^{2n-1}(BG; \mathbb{R}) = 0$, the Mayer-Vietoris sequence simplifies into pullback squares of the form (3.10). $\square$

Lemma 3.9 tells us that if we want to lift characteristic classes from $H^{2n}(BG; \mathbb{Z})$ to $H^{2n}(B_\bullet G; \mathbb{Z}(n))$, it suffices to understand the map $H^{2n}(B_\bullet G; \mathbb{R}(n)) \to H^{2n}(B_\bullet G; \mathbb{R})$. Note that we have a fiber sequence of sheaves

$$\Omega^k[-(k+1)] \longrightarrow \mathbb{R}(k+1) \xrightarrow{t} \mathbb{R}(k); \tag{3.11}$$

it will suffice to understand $H^i(B_\bullet G; \Omega^j)$. Bott computes this in [Bot73]:

Theorem 3.12 (Bott [Bot73]). *There is a natural ring isomorphism*

$$H^p(B_\bullet G; \Omega^q) \cong H^{p-q}_{sm}(G; \mathrm{Sym}^q(\mathfrak{g}^\vee)), \tag{3.13}$$

where $H^*_{sm}(G; \mathrm{Sym}^q(\mathfrak{g}^\vee))$ is the smooth cohomology *of G [HM62].*

Remark 3.14. We care about the case where $p = q = n$, where Theorem 3.12 states that

$$H^n(B_\bullet G; \Omega^n) \cong H^0_{sm}(G; \mathrm{Sym}^n(\mathfrak{g}^\vee)) = \mathrm{Sym}^n(\mathfrak{g}^\vee)^G = I^n(G). \tag{3.15}$$

Theorem 3.16 (Bott [Bot73]). *There is a natural isomorphism $\phi\colon I^n(G) \xrightarrow{\cong} H^{2n}(B_\bullet G; \mathbb{R}(n))$ such that the composition*

$$I^n(G) \xrightarrow{\phi} H^{2n}(B_\bullet G; \mathbb{R}(n)) \xrightarrow{t} H^{2n}(B_\bullet G; \mathbb{R}) = H^{2n}(BG; \mathbb{R}) \tag{3.17}$$

is the Chern-Weil homomorphism.

This is not exactly the same as the theorem Bott gave in [Bot73]; see [ADH21, Corollaries 16.2.4 and 16.2.5] for a proof of this version. By Chern-Weil theory, we know how to compute $H^{2n}(BG;\mathbb{R})$: let $j\colon K \hookrightarrow G$ be a maximal compact subgroup and $\mathfrak{k}$ be the Lie algebra of K. Then the composition

$$I^*(G) \xrightarrow{CW} H^{2n}(BG;\mathbb{R}) \xrightarrow{j^*} H^{2n}(BK;\mathbb{R}) \xrightarrow{CW^{-1}} I^*(K) \tag{3.18}$$

can be identified with the map $I^*(G) \to I^*(K)$ induced from the inclusion of Lie algebras $\mathfrak{k} \subset \mathfrak{g}$. Combine this with Lemma 3.9 to conclude:

Corollary 3.19. *Let G be a Lie group with $\pi_0(G)$ finite. let $j\colon K \hookrightarrow G$ be a maximal compact subgroup and $\mathfrak{k}$ be the Lie algebra of K. Then there is a pullback square*

$$\begin{tikzcd} H^{2n}(B_\bullet G;\mathbb{Z}(n)) \arrow[r, "t"] \arrow[d] & H^{2n}(BG;\mathbb{Z}) \arrow[d] \\ I^*(G) \arrow[r, "j^*"] & I^*(K), \end{tikzcd} \tag{3.20}$$

where t is truncation and j^* is the pullback of functions induced by the inclusion of Lie algebras $j : \mathfrak{k} \subset \mathfrak{g}$.

Corollary 3.21. *If G is a compact Lie group, the map $t\colon H^{2n}(B_\bullet G;\mathbb{Z}(n)) \to H^{2n}(BG;\mathbb{Z})$ is an isomorphism.*

That is, off-diagonal differential lifts of characteristic classes exist and are unique for compact Lie groups. For general noncompact G, off-diagonal differential lifts are not unique. We are interested in the case of $G = \mathrm{GL}_n^+(\mathbb{R})$, the group of orientation-preserving invertible $n \times n$ matrices. Characteristic classes for $\mathrm{GL}_n^+(\mathbb{R})$ correspond to characteristic classes of rank-n oriented real vector bundles.

Corollary 3.22 ([ADH21, §17.3])**.** *Let $n \geq 2$. The space of lifts of the first Pontryagin class $p_1 \in H^4(B\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z})$ across the truncation map*

$$t\colon H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}(2)) \longrightarrow H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}) = H^4(B\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}) \tag{3.23}$$

is a one-dimensional affine space. The image of these differential lifts under the map

$$H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}(2)) \to H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R});\mathbb{R}(2)) \xrightarrow{\cong} I^2(\mathrm{GL}_n^+(\mathbb{R})) \tag{3.24}$$

is the affine space $\{\lambda \mathrm{tr}(A)^2 - \frac{1}{8\pi^2}\mathrm{tr}(A^2) \mid \lambda \in \mathbb{R}\}$. Here, $\mathrm{tr} : \mathfrak{gl}_n^+(\mathbb{R}) \to \mathbb{R}$ is the usual trace map.

Proof. Take the maximal compact $\mathrm{SO}_n \subset \mathrm{GL}_n^+(\mathbb{R})$. The vector space $I^2(\mathrm{SO}_n)$ is one-dimensional, generated by $\mathrm{tr}(A^2)$. On the other hand, $I^2(\mathrm{GL}_n^+(\mathbb{R}))$ is two-dimensional, with basis $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A)^2$. The natural map

$$j^*\colon I^2(\mathrm{GL}_n^+(\mathbb{R})) \longrightarrow I^2(\mathrm{SO}_n) \tag{3.25}$$

is surjective with kernel $\mathrm{tr}(A)^2$, since the elements of $\mathfrak{so}_n$ are traceless. By Corollary 3.19, we have a pullback diagram

$$\begin{tikzcd} H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}(2)) \arrow[r, "t"] \arrow[d] & H^4(B\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}) \arrow[d] \\ I^2(\mathrm{GL}_n^+(\mathbb{R})) \arrow[r, "j^*"] & I^2(\mathrm{SO}_n). \end{tikzcd} \tag{3.26}$$

Recall that $p_1 \in H^4(B\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$ is a generator, and its image under the Chern-Weil homomorphism is $-\frac{1}{8\pi^2}\mathrm{tr}(A^2)$. Since j^* is surjective with a one-dimensional kernel, the set $(j^*)^{-1}(-\frac{1}{8\pi^2}\mathrm{tr}(A^2))$ is an affine line. Therefore $t^{-1}(p_1)$ is also an affine line, and its image in $I^2(\mathrm{GL}_n^+(\mathbb{R}))$ is precisely $(j^*)^{-1}(-\frac{1}{8\pi^2}(\mathrm{tr}(A^2)))$, which is the line of polynomials of the form $-\frac{1}{8\pi^2}\mathrm{tr}(A^2) + q$ for $q \in \ker(j^*)$. This is the affine line $\{\lambda \mathrm{tr}(A)^2 - \frac{1}{8\pi^2}\mathrm{tr}(A^2) \mid \lambda \in \mathbb{R}\}$. □

When $n = 1$, $\mathrm{GL}_1^+(\mathbb{R})$ is homotopy equivalent to a point and $B\mathrm{GL}_1^+(\mathbb{R}) \simeq *$. Therefore $H^4(B\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}) \simeq H^4(*;\mathbb{Z}) \simeq 0$. The first Pontryagin class is $0 \in H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z})$. On the other hand, the stack $B_\bullet\mathrm{GL}_1^+(\mathbb{R})$ is nontrivial. We declare that the differential lifts of the first Pontryagin class are the preimages of 0 under the map

$$H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2)) \to H^4(B\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}) = 0, \tag{3.27}$$

that is, all of $H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2))$.

Corollary 3.28. *For $\mathrm{GL}_1^+(\mathbb{R})$, $H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2))$ is a one-dimensional $\mathbb{R}$ vector space, so there is a one-dimensional vector space of lifts of the first Pontryagin class. The image of this vector space under the map*

$$H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2)) \to H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{R}(2)) \xrightarrow{\cong} I^2(\mathrm{GL}_1^+(\mathbb{R})) \tag{3.29}$$

is the one-dimensional vector space $I^2(\mathrm{GL}_1^+(\mathbb{R})) = \{\lambda A^2 \mid \lambda \in \mathbb{R}\}$.

Proof. Applying Corollary 3.19 to $\mathrm{GL}_1^+(\mathbb{R})$, we see that

$$H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z}(2)) \longrightarrow I^2(\mathrm{GL}_1^+(\mathbb{R})) \tag{3.30}$$

is an isomorphism. As $I^2(\mathrm{GL}_1^+(\mathbb{R}))$ is a one-dimensional vector space generated by the polynomial A^2, we have a whole line of differential lifts of the trivial first Pontryagin class $p_1 = 0 \in H^4(B\mathrm{GL}_1^+(\mathbb{R});\mathbb{Z})$. □

Remark 3.31. The first Pontryagin classes are defined for all $\mathrm{GL}_n^+(\mathbb{R})$ for $n \geq 1$. By abuse of notation we denote all of them by p_1. They form a compatible family of characteristic classes in the following sense: for $n \leq m$, consider the inclusion $\mathrm{GL}_n^+(\mathbb{R}) \subset \mathrm{GL}_m^+(\mathbb{R})$ coming from the inclusion $\mathbb{R}^n \subset \mathbb{R}^m$ as the first n coordinates; then the pullback of $p_1 \in H^4(B\mathrm{GL}_m^+(\mathbb{R});\mathbb{Z})$ along the map

$$H^4(B\mathrm{GL}_m^+(\mathbb{R});\mathbb{Z}) \longrightarrow H^4(B\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z}) \tag{3.32}$$

is the first Pontryagin class $p_1 \in H^4(B\mathrm{GL}_n^+(\mathbb{R});\mathbb{Z})$. Alternatively, we can think about p_1 as a class in $H^4(B\mathrm{GL}_\infty^+(\mathbb{R});\mathbb{Z})$.

This family of classes satisfies the *Whitney sum formula*: for all oriented real vector bundles $E, F \to X$,

$$p_1(E \oplus F) = p_1(E) + p_1(F) \tag{3.33}$$

in $H^4(X;\mathbb{Z})$. Here there is an important subtlety: in general, the Whitney sum formula for Pontryagin classes only holds modulo 2-torsion. But for orientable vector bundles, (3.33) does hold integrally: Brown [Bro82, Theorem 1.6] showed that for arbitrary vector bundles, the difference $p_1(E \oplus F) - p_1(E) - p_1(F)$ can be expressed in terms of $w_1(E)$ and $w_1(F)$ (see also [Tho62]), so the difference vanishes when E and F are orientable. So it is crucial that we are working with $\mathrm{GL}_n^+(\mathbb{R})$, and with p_1 — even for oriented vector bundles, the Whitney sum formula does not hold in general for higher-degree Pontryagin classes.

The differential lifts of the first Pontryagin class also form a one-dimensional affine family of compatible classes labeled by $\lambda \in \mathbb{R}$: given λ, the compatible classes $\hat{p}_1^\lambda \in H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R}); \mathbb{Z}(2))$ are defined as follows:

(1) For $n \geq 2$, the image of $\hat{p}_1^\lambda$ under the map

$$H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R}); \mathbb{Z}(2)) \longrightarrow I^2(\mathrm{GL}_n^+(\mathbb{R})) \tag{3.34}$$

is $\lambda\mathrm{tr}(A)^2 - \frac{1}{8\pi^2}\mathrm{tr}(A^2)$.

(2) For $n = 1$, the image of $\hat{p}_1^\lambda$ under the map

$$H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2)) \longrightarrow I^2(\mathrm{GL}_1^+(\mathbb{R})) \tag{3.35}$$

is $(\lambda - \frac{1}{8\pi^2})A^2$.

This is a compatible family in the sense of Remark 3.31: these classes pull back to one another under the maps $B_\bullet\mathrm{GL}_m^+(\mathbb{R}) \to B_\bullet\mathrm{GL}_n^+(\mathbb{R})$. Both $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A)^2$ in $H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R}); \mathbb{Z}(2))$ pull back to A^2 in $H^4(B_\bullet\mathrm{GL}_1^+(\mathbb{R}); \mathbb{Z}(2))$.

We can ask for a family of lifts $\hat{p}_1^\lambda$ to satisfy the *differential Whitney sum formula*: that for all oriented real vector bundles $E, F \to X$, the equation

$$\hat{p}_1^\lambda(E \oplus F) = \hat{p}_1^\lambda(E) + \hat{p}_1^\lambda(F) \tag{3.36}$$

holds in $H^4(X; \mathbb{Z}(2))$.

Lemma 3.37. *There is a unique family of lifts of p_1 that satisfy (3.36). This family is the $\lambda = 0$ family, corresponding to the Chern-Weil forms*

$$-\frac{1}{8\pi^2}\mathrm{tr}(A^2) \in I^2(\mathrm{GL}_n^+(\mathbb{R})) \tag{3.38}$$

for $n \geq 2$ and

$$-\frac{1}{8\pi^2}A^2 \in I^2(\mathrm{GL}_1^+(\mathbb{R})) \tag{3.39}$$

for $n = 1$.

Proof. It is sufficient to check the universal case, where $X = B_\bullet\mathrm{GL}_n^+(\mathbb{R}) \times B_\bullet\mathrm{GL}_m^+(\mathbb{R})$ and E and F are the tautological bundles on the first, resp. second factors. We denote the projections to the factors by $\mathrm{pr}_n, \mathrm{pr}_m \colon X \to B_\bullet\mathrm{GL}_{n,m}^+(\mathbb{R})$. Direct sum of vector bundles induces a map

$$s\colon B_\bullet\mathrm{GL}_n^+(\mathbb{R}) \times B_\bullet\mathrm{GL}_m^+(\mathbb{R}) \to B_\bullet\mathrm{GL}_{n+m}^+(\mathbb{R}). \tag{3.40}$$

The differential Whitney sum formula (3.36) reads

$$\mathrm{pr}_1^*(\hat{p}_1^\lambda) + \mathrm{pr}_2^*(\hat{p}_1^\lambda) = s^*(\hat{p}_1^\lambda). \tag{3.41}$$

We can check this under the injective map

$$t : H^4(B_\bullet\mathrm{GL}_n^+(\mathbb{R}) \times B_\bullet\mathrm{GL}_m^+(\mathbb{R}); \mathbb{Z}(2)) \to I^2(\mathrm{GL}_m^+(\mathbb{R}) \times \mathrm{GL}_n^+(\mathbb{R})). \tag{3.42}$$

We interpret an element $A \in \mathfrak{gl}_n^+(\mathbb{R}) \oplus \mathfrak{gl}_m^+(\mathbb{R})$ as a block diagonal matrix $A_n \oplus A_m \in \mathfrak{gl}_{n+m}^+(\mathbb{R})$. Plugging in the polynomials corresponding to $\hat{p}_1^\lambda$ (see Corollary 3.22), (3.41) becomes

$$\lambda\left(\mathrm{tr}(A_n)^2 + \mathrm{tr}(A_m)^2\right) - \frac{1}{8\pi^2}\left(\mathrm{tr}(A_n^2) + \mathrm{tr}(A_m^2)\right) = \lambda\mathrm{tr}(A_n \oplus A_m)^2 - \frac{1}{8\pi^2}\mathrm{tr}((A_n \oplus A_m)^2), \tag{3.43}$$

which forces $\lambda = 0$. $\square$

We will denote this distinguished family by $\hat{p}_1$.

4. Cocycles for off-diagonal differential lifts

In this section we compute explicit cocycles for the differential lifts of the first Pontryagin class for $G = \mathrm{GL}_1^+(\mathbb{R}) = \mathbb{R}_+^\times$. We will identify $\mathbb{R}_+^\times$ with $\mathbb{R}$ via the natural logarithm, inverse to $\exp\colon \mathbb{R} \to \mathbb{R}_+^\times$.

Recall from Section 4 that that there is a one-dimensional vector space of lifts of p_1 for $\mathrm{GL}_1^+(\mathbb{R})$ (Corollary 3.28), with a distinguished element $\hat{p}_1$ whose image under the isomorphism $H^4(B_\bullet G; \mathbb{Z}(2)) \to I^2(G)$ is $-\frac{1}{8\pi^2}A^2$ (Lemma 3.37). We would like to find a cocycle representing $\hat{p}_1 \in H^4(B_\bullet\mathbb{R}; \mathbb{Z}(2))$; by scaling, this will give us cocycle representatives for the entire line of differential lifts of p_1.

If M is a paracompact manifold, possibly infinite-dimensional, then M admits a partition of unity, so Ω^i is acyclic on M. This implies the following well-known lemma:

Lemma 4.1. *Let G be a paracompact Lie group, possibly infinite-dimensional. Then the cohomology groups $H^*(B_\bullet G; \Omega^i)$ are computed by the cochain complex*

$$\Omega^i(B_\bullet G) : \Omega^i(*) \longrightarrow \Omega^1(G) \longrightarrow \Omega^i(G\times G) \longrightarrow \Omega^i(G\times G\times G)\dots. \tag{4.2}$$

All finite-dimensional Lie groups are paracompact, and so is Γ [Pal21, Remark 4.11].

Consider the following factorization of the exterior derivative $\mathrm{d} : \Omega^1 \to \Omega^2$:

$$\Omega^1 \xrightarrow{\iota} \mathbb{Z}(2)[2] \xrightarrow{\mathrm{d}_1} \Omega^2, \tag{4.3}$$

where ι is the inclusion of Ω^1 into the complex $\mathbb{Z}(2)[2]$, and d_1 is d on Ω^1 and 0 on the other terms in $\mathbb{Z}(2)$. We will also abuse notation and let d, d_1, and ι denote the maps on cohomology induced by these maps between complexes of sheaves.

Lemma 4.4. *The maps of sheaves in (4.3) induce a triangle of isomorphisms between three real lines*

$$\begin{tikzcd} & H^4(B_\bullet\mathbb{R};\mathbb{Z}(2)) \arrow[dr, "\mathrm{d}_1"] & \\ H^2(B_\bullet\mathbb{R};\Omega^1) \arrow[ur, "\iota"] \arrow[rr, "\mathrm{d}"] & & H^2(B_\bullet\mathbb{R};\Omega^2). \end{tikzcd} \tag{4.5}$$

Proof. It suffices to prove two of the above maps are isomorphisms and that one of the groups is abstractly isomorphic to $\mathbb{R}$. We first show ι is an isomorphism and then show d is a nonzero map $\mathbb{R} \to \mathbb{R}$.

We begin by proving $H^k(B_\bullet\mathbb{R}; \mathbb{Z}(1)) = 0$ for $k > 2$. This follows from

$$H^k(B_\bullet\mathbb{R};\mathbb{Z}) = H^k(B\mathbb{R};\mathbb{Z}) = H^k(*;\mathbb{Z}) \overset{k\geq 0}{=} 0,$$
$$H^k(B_\bullet\mathbb{R};\Omega^0) = H^k_{\mathrm{Lie}}(\mathfrak{g}_\mathbb{R};\mathbb{R}) \cong H^k_{\mathrm{Lie}}(\mathbb{R};\mathbb{R}) \overset{k\geq 1}{=} 0,$$

and the long exact sequence

$$\cdots \to H^k(B_\bullet\mathbb{R};\Omega^0) \to H^{k+1}(B_\bullet\mathbb{R};\mathbb{Z}(1)) \to H^{k+1}(B_\bullet\mathbb{R};\mathbb{Z}) \to H^{k+1}(B_\bullet\mathbb{R};\Omega^0) \to \cdots. \tag{4.6}$$

Now consider the fiber sequence of coefficient sheaves

$$\Omega^1 \xrightarrow{\iota} \mathbb{Z}(2)[2] \longrightarrow \mathbb{Z}(1)[2]. \tag{4.7}$$

The associated long exact sequence in cohomology shows

$$\iota : H^2(B_\bullet\mathbb{R};\Omega^1) \xrightarrow{\simeq} H^2(B_\bullet\mathbb{R};\mathbb{Z}(2)[2]) = H^4(B_\bullet\mathbb{R};\mathbb{Z}(2)) \tag{4.8}$$

is an isomorphism, because both $H^1(B_\bullet\mathbb{R};\mathbb{Z}(1)[2]) = H^3(B_\bullet\mathbb{R};\mathbb{Z}(1))$ and $H^2(B_\bullet\mathbb{R};\mathbb{Z}(1)[2]) = H^4(B_\bullet\mathbb{R};\mathbb{Z}(1))$ vanish.

Using the explicit model for $H^2(B_\bullet\mathbb{R};\Omega^i)$ given in Lemma 4.1, it is straightforward to show that $x_1\,\mathrm{d}x_2 \in \Omega^1(\mathbb{R}^2)$ is a nontrivial Ω^1-valued 2-cocycle over $B_\bullet\mathbb{R}$ Similarly, its differential $\mathrm{d}x_1\,\mathrm{d}x_2 \in \Omega^2(\mathbb{R}^2)$ is nontrivial in $H^2(B_\bullet\mathbb{R};\Omega^2)$. Denote the Lie algebra of $\mathbb{R}$ by $\mathfrak{g}_\mathbb{R} \cong \mathbb{R}$. Bott's theorem (Theorem 3.12) implies

$$H^2(B_\bullet\mathbb{R};\Omega^1) \cong H^1_{sm}(\mathbb{R};\mathfrak{g}_\mathbb{R}^\vee) \cong \mathrm{Hom}(\mathbb{R},\mathbb{R}) = \mathbb{R} \tag{4.9}$$

$$H^2(B_\bullet\mathbb{R};\Omega^2) \cong H^0_{sm}(\mathbb{R};\mathrm{Sym}^2\mathfrak{g}_\mathbb{R}^\vee) = (\mathrm{Sym}^2(\mathfrak{g}_\mathbb{R}^\vee))^\mathbb{R} = \mathrm{Sym}^2(\mathbb{R}) \cong \mathbb{R}, \tag{4.10}$$

so the cocycles $[x_1\,\mathrm{d}x_2] \in H^2(B_\bullet\mathbb{R};\Omega^1)$ and its image under d, $[\,\mathrm{d}x_1\,\mathrm{d}x_2] \in H^2(B_\bullet\mathbb{R};\Omega^2)$, generate the cohomology groups they live in. This shows that indeed, d is a nonzero map of real lines. ☐

Proposition 4.11. *The preimage of the distinguished class $\hat{p}_1 \in H^4(B_\bullet\mathbb{R};\mathbb{Z}(2))$ under the isomorphism $\iota : H^2(B_\bullet\mathbb{R};\Omega^1) \xrightarrow{\simeq} H^4(B_\bullet\mathbb{R};\mathbb{Z}(2))$ from Lemma 4.4 has a cocycle representative $\frac{1}{8\pi^2}x_1\,\mathrm{d}x_2$ in $\Omega^1(\mathbb{R}^2)$.*

Proof. The proof boils down to computing a proportionality factor coming from the standard normalization of Chern-Weil theory. Consider diagram (4.12): in Lemma 4.4, we saw that $\mathrm{d}\colon H^2(B_\bullet\mathbb{R};\Omega^1) \to H^2(B_\bullet\mathbb{R};\Omega^2)$ factors as $\mathrm{d}_1 \circ \iota$, and in Theorems 3.12 and 3.16, we saw that $\mathrm{d}_1\colon H^4(B_\bullet\mathbb{R};\mathbb{Z}(2)) \to H^2(B_\bullet\mathbb{R};\Omega^2)$ further factors through $H^4(B_\bullet\mathbb{R};\mathbb{R}(2))$ and $I^2(\mathbb{R})$.

(4.12)

$$\begin{tikzcd}
H^2(B_\bullet\mathbb{R};\Omega^1) \arrow[r, "\iota"] \arrow[rrrr, bend left, "\mathrm{d}"] & H^4(B_\bullet\mathbb{R};\mathbb{Z}(2)) \arrow[r] & H^4(B_\bullet\mathbb{R};\mathbb{R}(2)) \arrow[r, "(3.16)"'] & I^2(\mathbb{R}) \arrow[r, "(3.15)"'] & H^2(B_\bullet\mathbb{R};\Omega^2) \\
\frac{1}{8\pi^2}x_1\,\mathrm{d}x_2 \arrow[rrrr, mapsto] & & & & \frac{1}{8\pi^2}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
& \hat{p}_1 \arrow[rr, mapsto] & & -\frac{1}{8\pi^2}A^2 \arrow[r, mapsto, "?"] & \frac{1}{8\pi^2}\,\mathrm{d}x_1\,\mathrm{d}x_2
\end{tikzcd}$$

It thus suffices to show that the Bott isomorphism (3.15) indeed sends $-\frac{1}{8\pi^2}A^2 \mapsto \frac{1}{8\pi^2}\,\mathrm{d}x_1\,\mathrm{d}x_2$. In that case, commutativity of the maps in diagram (4.12) implies that ι maps $-\frac{1}{8\pi^2}A^2 \mapsto \hat{p}_1$. The Bott isomorphism is normalized to match Chern-Weil theory (see Theorem 3.16). The maps fit together as indicated in diagram (4.13), which is natural in G.

$$\begin{tikzcd}
H^{2n}(B_\bullet G;\mathbb{R}(n)) \arrow[r] \arrow[d] & H^n(B_\bullet G;\Omega^n) \\
H^{2n}(BG;\mathbb{R}) & I^n(G) \arrow[l, "CW"] \arrow[ul, "\phi"'] \arrow[u, "Bott\ (3.15)"']
\end{tikzcd} \tag{4.13}$$

It suffices to consider $G = \mathrm{U}_1$. We determine the normalization on the first Chern class c_1 in cohomological degree 2. Then we use multiplicativity of the Chern-Weil and Bott maps to deduce the normalization in degree 4, where $\hat{p}_1$ lives. The groups containing $c_1 \in H^2(B\mathrm{U}_1;\mathbb{Z})$ and its unique lift (provided by Corollary 3.21) $\hat{c}_1 \in H^2(B_\bullet\mathrm{U}_1;\mathbb{Z}(1))$ fit into the relevant diagram as follows:

$$
\begin{tikzcd}
H^2(B_\bullet \mathrm{U}_1;\mathbb{Z}(1)) \arrow[r, hook] \arrow[d, "\cong"] & H^2(B_\bullet \mathrm{U}_1;\mathbb{R}(1)) \arrow[r] \arrow[d, "\cong"] & H^1(B_\bullet \mathrm{U}_1;\Omega^1) \\
H^2(B\mathrm{U}_1;\mathbb{Z}) \arrow[r, hook] & H^2(B\mathrm{U}_1;\mathbb{R}) & I^1(\mathrm{U}_1). \arrow[l, "CW"] \arrow[ul, "\phi"'] \arrow[u, "Bott\ (3.15)"']
\end{tikzcd}
\tag{4.14}
$$

We identify the Lie algebra of U_1 as $\mathfrak{u}_1 = i\mathbb{R}$ via the exponential map $\exp(2\pi i -)\colon i\mathbb{R} \to \mathrm{U}_1$. Chern-Weil theory associates to c_1 the polynomial $\frac{1}{2\pi i}\mathrm{id}\colon i\mathbb{R} \to \mathbb{R}$. The class c_1 generates $H^2(B\mathrm{U}_1;\mathbb{Z})$, and thus its lift $\hat{c}_1$ generates $H^2(B\mathrm{U}_1;\mathbb{Z}(1)) \simeq H^1(B\mathrm{U}_1;\mathbb{T})$. The group $H^1(B\mathrm{U}_1;\mathbb{T})$ is the group of homomorphisms $\mathrm{U}_1 \to \mathbb{T}$ (this is a general fact about Lie group cohomology, see e.g. [WW15]). Under this identification, $\hat{c}_1$ corresponds to the homomorphism $\chi_1\colon \mathrm{U}_1 \xrightarrow{\mathrm{id}} \mathrm{U}_1 = \mathbb{T}$.

The composition of maps in the top row of (4.14) is induced by the map of sheaves $\mathrm{d}_0\colon \mathbb{Z}(1) \to \Omega^1[-1]$, given by d on the Ω^0-term in $\mathbb{Z}(1)$, and 0 on the other terms. This map factors through the sheaf $\mathbb{T}$ as

$$
\mathbb{Z}(1) \xrightarrow{\exp(2\pi i -)} \mathbb{T}[-1] \xrightarrow{\frac{1}{2\pi i}\,\mathrm{d}\log(-)} \Omega^1[-1]. \tag{4.15}
$$

The image of $\hat{c}_1$ in $H^1(B_\bullet \mathrm{U}_1;\Omega^1)$ is the image of χ_1 under the latter map. We pick a representing 1-form

$$
\mathrm{d}\theta \coloneqq \frac{1}{2\pi i}\,\mathrm{d}\log(\chi_1) \in \Omega^1(\mathrm{U}_1). \tag{4.16}
$$

Now we use naturality of the Bott isomorphism to conclude the normalization for the group $\mathbb{R}$. We identify the Lie algebra of $\mathbb{R}$ as $\mathfrak{g}_\mathbb{R} = \mathbb{R}$, with exponential given by the identity map $\mathrm{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}$. The covering map $\exp(2\pi i -) : \mathbb{R} \to \mathrm{U}_1$ induces the map $2\pi i$ on Lie algebras: $\mathrm{d}_e \exp(2\pi i -) : \mathfrak{g}_\mathbb{R} = \mathbb{R} \xrightarrow{2\pi i} i\mathbb{R} = \mathfrak{u}_1$. As a result, the induced map on the degree-one polynomials on the Lie algebras is the transpose of this linear map, $2\pi i : \mathfrak{u}_1^\vee = i\mathbb{R} \to \mathbb{R} = \mathfrak{g}_\mathbb{R}^\vee$. This is the left-hand map in the naturality square for the Bott isomorphism:

$$
\begin{tikzcd}
I^1(\mathrm{U}_1) = i\mathbb{R}^\vee = i\mathbb{R} \arrow[r, "Bott"] \arrow[d] & H^1(B_\bullet \mathrm{U}_1;\Omega^1) \arrow[d] \\
I^1(\mathbb{R}) = \mathbb{R}^\vee = \mathbb{R} \arrow[r, "Bott"] & H^1(B_\bullet \mathbb{R};\Omega^1).
\end{tikzcd}
\tag{4.17}
$$

We compute the right-hand map $H^1(B_\bullet \mathrm{U}_1;\Omega^1) \to H^1(B_\bullet \mathbb{R};\Omega^1)$ by pulling back the representing differential forms. The form $\mathrm{d}\theta \in \Omega^1(\mathrm{U}_1)$ pulls back to the differential form $\mathrm{d}x \in \Omega^1(\mathbb{R})$, the unique right-invariant differential form on $\mathbb{R}$ whose restriction to the Lie algebra is the identity functional $\mathrm{d}x_e : \mathfrak{g}_\mathbb{R} = \mathbb{R} \xrightarrow{\mathrm{id}_\mathbb{R}} \mathbb{R}$. The preimage of $\mathrm{d}\theta$ under the Bott isomorphism is the linear functional $\frac{1}{2\pi i}\,\mathrm{d}\theta_e : \mathfrak{u}_1 = i\mathbb{R} \xrightarrow{\frac{1}{2\pi i}} \mathbb{R}$. In summary,

$$
\begin{tikzcd}
\frac{1}{2\pi i}\,\mathrm{d}\theta_e \arrow[r, mapsto, "Bott"] \arrow[d, mapsto] & \mathrm{d}\theta \arrow[d, mapsto] \\
\mathrm{d}x_e \arrow[r, mapsto, "Bott"] & \mathrm{d}x,
\end{tikzcd}
\tag{4.18}
$$

and so the linear functional $\mathrm{d}x_e : \mathfrak{g}_\mathbb{R} \to \mathbb{R}$ is sent to to $\mathrm{d}x \in \Omega^1(\mathbb{R})$. Since the Bott isomorphism is multiplicative, we can calculate the image of $\mathrm{d}x^2$ using the cup product structure on the bigraded ring $H^*(B_\bullet \mathbb{R}, \Omega^*)$.[12] This is the classic formula for the cup product of Čech cochains with values

[12] To clarify, the two gradings are H^* and Ω^*; the simplicial direction in $B_\bullet \mathbb{R}$ does not come into play here.

in a complex of sheaves with ring structure. In our case, the Čech cover is the map $* \to */\mathbb{R}$ (with Čech nerve the simplicial manifold $B_\bullet\mathbb{R}$), and the complex of sheaves is $\Omega^\bullet$ (with ring structure given by the wedge of differential forms). The cochain $\mathrm{d}x \in \Omega^1(\mathbb{R})$ lives in bidegree $(1,1)$. Its square $\mathrm{d}x^2$ has bidegree $(2,2)$ and is given by $-p_1^*(\mathrm{d}x)p_2^*(\mathrm{d}x) = -\mathrm{d}x_1\,\mathrm{d}x_2 \in \Omega^2(\mathbb{R}\times\mathbb{R})$ (see [Sta22, Tag 01FP] or [AGV71, Exposé XVII] for the definition of the cup product). The distinguished Pontryagin class thus maps to $-\frac{1}{8\pi^2}\,\mathrm{d}{x_e}^2 \mapsto \frac{1}{8\pi^2}\,\mathrm{d}x_1\,\mathrm{d}x_2 \in \Omega^2(\mathbb{R}^2)$. □

5. Computing the transgression

Recall $\mathrm{GL}_1^+(\mathbb{R}) = \mathbb{R}_+^\times$ and $\Gamma = \mathrm{Diff}^+(S^1)$. We identify $\mathrm{GL}_1^+(\mathbb{R})$ with $\mathbb{R}$ via the isomorphism $\log : \mathbb{R}_+^\times \to \mathbb{R}$. Let $F = \mathrm{Fr}^+(S^1) \to S^1$ be the oriented frame bundle of S^1. This is a trivial $\mathrm{GL}_1^+(\mathbb{R})$-bundle: it is canonically identified with $S^1 \times \mathbb{R}_+^\times \to S^1$. The action of Γ on S^1 is orientation-preserving, and thus lifts to an action on F. This Γ-action commutes with the $\mathbb{R}$-action on F. We think of Γ as acting on F on the left and $\mathbb{R}$ as acting on the right. The double quotient stack $\Gamma\backslash F/\mathbb{R}$ fits into the diagram

$$\begin{tikzcd} \Gamma\backslash F/\mathbb{R} \arrow[r, "q"] \arrow[d, "p"] & B_\bullet\mathbb{R} \\ B_\bullet\Gamma. \end{tikzcd} \tag{5.1}$$

Note that $F/\mathbb{R} = S^1$. Under this identification, $\Gamma\backslash F/\mathbb{R} = \Gamma\backslash S^1 \to B_\bullet\Gamma$ is the tautological oriented S^1-bundle over $B_\bullet\Gamma$. We will compute the transgression map $p_*q^* : H^4(B_\bullet\mathbb{R};\mathbb{Z}(2)) \to H^3(B_\bullet\Gamma;\mathbb{Z}(1))$ associated to (5.1). The pushforward p_* is fiber integration in an S^1-bundle, so we denote it by $\int_{S^1}$.

In §3, we constructed differential lifts of the first Pontryagin class in $H^4(B_\bullet\mathbb{R};\mathbb{Z}(2))$, and in Lemma 2.6, we saw that $H^3(B_\bullet\Gamma;\mathbb{Z}(1)) = H^2(B_\bullet\Gamma;\underline{\mathbb{T}})$ classifies Fréchet Lie group central extensions of Γ by $\mathbb{T}$. Therefore applying this transgression map turns a differential lift of p_1 into a central extension of Γ by $\mathbb{T}$.

Theorem 5.2. *The transgression homomorphism*

$$\int_{S^1} \circ\, q^* : H^4(B_\bullet\mathbb{R};\mathbb{Z}(2)) \longrightarrow H^2(B_\bullet\Gamma;\mathbb{Z}(1)) \tag{5.3}$$

lands isomorphically in the Virasoro family $\mathcal{V}ir \subset \mathrm{CExt}_{\mathbb{R}}(\Gamma) \cong H^2(B_\bullet\Gamma;\mathbb{Z}(1))$. It induces an isomorphism

$$I^2(\mathbb{R}) \longrightarrow \mathcal{V}ir \tag{5.4}$$

and takes the distinguished first Pontryagin class $\hat{p}_1$ (Lemma 3.37) to the Virasoro central extension $\widetilde{\Gamma}_{-12}$. That is, the Virasoro group obtained by transgressing $\hat{p}_1$ has central charge -12.

Remark 5.5. Recall from Remark 1.6 that we choose our normalization so that $\widetilde{\Gamma}_1$ has central charge 1, that is, it is the central extension that acts on the bosonic string CFT.

We begin by reducing to a computation on differential forms. By Proposition 4.11, the map $H^2(B_\bullet\mathbb{R},\Omega^1) \to H^4(B_\bullet\mathbb{R};\mathbb{Z}(2))$ is an isomorphism, and the preimage of $\hat{p}_1$ is $\frac{1}{8\pi^2}x_1\,\mathrm{d}x_2 \in \Omega^1(\mathbb{R}^2)$.

Functoriality of pullback and integration imply the following commutative diagram (Lemma 2.16):

$$\begin{tikzcd} H^2(B_\bullet\mathbb{R};\Omega^1) \arrow[r, "q^*"] \arrow[d, "\cong"] & H^2(\Gamma\backslash F/G;\Omega^1) \arrow[r, "\int_{S^1}"] \arrow[d] & H^2(B_\bullet\Gamma;\Omega^0) \arrow[d, "\phi"] \\ H^4(B_\bullet\mathbb{R};\mathbb{Z}(2)) \arrow[r, "q^*"] & H^4(\Gamma\backslash F/G;\mathbb{Z}(2)) \arrow[r, "\int_{S^1}"] & H^3(B_\bullet\Gamma;\mathbb{Z}(1)). \end{tikzcd} \tag{5.6}$$

The composition

$$H^2(B_\bullet\Gamma;\Omega^0) \xrightarrow{\phi} H^3(B_\bullet\Gamma;\mathbb{Z}(1)) \xrightarrow{\cong} H^2(B_\bullet\Gamma;\underline{\mathbb{T}}) \tag{5.7}$$

is the map induced by

$$\exp(2\pi i -) : \Omega^0 = \underline{\mathbb{R}} \to \underline{\mathbb{T}}. \tag{5.8}$$

Therefore we can compute the transgression using the top line of the diagram (5.6), where we have the advantage of working with differential forms, and then exponentiate to get back to $H^3(B_\bullet\Gamma;\mathbb{Z}(1))$.[13] In particular, to prove Theorem 5.2, it suffices to show the following:

Proposition 5.9. *The transgression map* $\int_{S^1} \circ\, q^* : H^2(B_\bullet\mathbb{R};\Omega^1) \to H^2(B_\bullet\Gamma;\Omega^0)$ *maps the class* $[x_1\,\mathrm{d}x_2]$ *to the class of the central extension* $\mathbb{R} \to \widetilde{\Gamma}_\mathbb{R} \to \Gamma$ *corresponding to the unnormalized* $\mathbb{R}$*-valued Bott-Thurston cocycle*[14]

$$B_\mathbb{R}(\gamma_1,\gamma_2) \coloneqq \int_{S^1} \log(\gamma_1' \circ \gamma_2)\,\mathrm{d}(\log(\gamma_2')). \tag{5.10}$$

We compute the map explicitly on cocycles, using simplicial presentations of the spaces and stacks involved. Recall that elements of $\Omega^1(\mathbb{R}^2)$ are cochains for $H^2(B_\bullet\mathbb{R};\Omega^1)$ with respect to the following simplicial presentation of $B_\bullet\mathbb{R}$:

$$* \Lleftarrow \mathbb{R} \Lleftarrow \mathbb{R}\times\mathbb{R} \Lleftarrow \mathbb{R}\times\mathbb{R}\times\mathbb{R}\cdots. \tag{5.11}$$

We compute the pullback of $x_1\,\mathrm{d}x_2 \in \Omega^1(\mathbb{R}^2)$ to $\Gamma\backslash F/\mathbb{R}$ using the following simplicial presentation of the map $\Gamma\backslash F/\mathbb{R} \to B_\bullet\mathbb{R}$:

$$\begin{tikzcd} \Gamma\backslash F \arrow[d] & \Gamma\backslash F\times\mathbb{R} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & \Gamma\backslash F\times\mathbb{R}\times\mathbb{R} \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \arrow[d] & \Gamma\backslash F\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\dots \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \arrow[d] \\ * & \mathbb{R} \arrow[l, shift left] \arrow[l, shift right] & \mathbb{R}\times\mathbb{R} \arrow[l, shift left] \arrow[l] \arrow[l, shift right] & \mathbb{R}\times\mathbb{R}\times\mathbb{R}\dots \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \end{tikzcd} \tag{5.12}$$

where the vertical maps simply forget the first factor. As a result, the image of $x_1\,\mathrm{d}x_2$ under the pullback map $p^* : \Omega^1(B_\bullet\mathbb{R}) \to \Omega^1((\Gamma\backslash F)/\mathbb{R}_\bullet)$ is constant along the factor $\Gamma\backslash F$. We denote the cocycle $p^*(x_1\,\mathrm{d}x_2)$ by the same symbol, $x_1\,\mathrm{d}x_2 \in \Omega^1(\Gamma\backslash F\times\mathbb{R}^2)$.

To compute the pushforward of this class along the projection $\Gamma\backslash F/\mathbb{R} \to B_\bullet\Gamma$, we pick a different presentation of $\Gamma\backslash F/\mathbb{R}$. Instead of resolving the $\mathbb{R}$-action, we resolve the Γ-action:

$$F/\mathbb{R} \Lleftarrow \Gamma\times F/\mathbb{R} \Lleftarrow \Gamma\times\Gamma\times F/\mathbb{R} \Lleftarrow \Gamma\times\Gamma\times\Gamma\times F/\mathbb{R}\dots. \tag{5.13}$$

[13]This last step is a map $H^2(B_\bullet\Gamma;\underline{\mathbb{R}}) \to H^2(B_\bullet\Gamma;\underline{\mathbb{T}})$ and thus can be interpreted as taking a central extension of Γ by $\mathbb{R}$ and building a central extension by $\mathbb{T}$.

[14]This "unnormalized" cocycle corresponds to the case $\lambda = -96\pi^2$ in (1.9).

The map $q\colon \Gamma\backslash F/\mathbb{R} \to B_\bullet\Gamma$ admits a presentation by the simplicial map

$$\begin{array}{ccccccc} F/\mathbb{R} & \leftleftarrows & \Gamma \times F/\mathbb{R} & \Lleftarrow & \Gamma\times\Gamma\times F/\mathbb{R} & \Lleftarrow & \Gamma\times\Gamma\times\Gamma\times F/\mathbb{R}\dots. \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \leftleftarrows & \Gamma & \Lleftarrow & \Gamma\times\Gamma & \Lleftarrow & \Gamma\times\Gamma\times\Gamma\dots \end{array} \tag{5.14}$$

which is a level-wise S^1-fibration. As a result, $p_*\colon \Omega^1(\Gamma\backslash(F/\mathbb{R})_\bullet) \to \Omega^1(B_\bullet\Gamma)$ may be computed by integrating over S^1 level-by-level.

In Lemma 5.22, we will prove that $x_1\,\mathrm{d}x_2 \in \Omega^1(\Gamma\backslash F\times\mathbb{R}^2)$ is cohomologous to the cocycle given by the integrand of the $\mathbb{R}$-valued Bott-Thurston cocycle (5.10), $\log(\gamma_1'\circ\gamma_2)\,\mathrm{d}(\log(\gamma_2'))$. This is enough to imply Proposition 5.9: we saw above using (5.12) that $x_1\,\mathrm{d}x_2 \in \Omega^1(\Gamma\backslash(F/\mathbb{R})_\bullet)$ represents the cohomology class $q^*([x_1\,\mathrm{d}x_2])$. Lemma 5.22 will show that $\log(\gamma_1'\circ\gamma_2)\,\mathrm{d}(\log(\gamma_2'))$ represents the same cohomology class; thus we can use the latter cocycle to compute the pushforward, and using (5.14), the pushforward will be $\int \log(\gamma_1'\circ\gamma_2)\,\mathrm{d}(\log(\gamma_2'))$, proving Proposition 5.9. So all we have left to do is prove Lemma 5.22.

The challenge is transporting our cocycle from the first simplicial presentation of $\Gamma\backslash F/\mathbb{R}$ (where we resolve in the $\mathbb{R}$-direction) to the second (where we resolve in the Γ-direction). To do so, we will chase it across the double complex associated to the bisimplicial manifold $\Gamma\backslash F/\mathbb{R}_{\bullet,\bullet}$ obtained by resolving both of these objects. Specifically, $\Gamma\backslash F/\mathbb{R}_{p,q} = \Gamma^p\times F\times\mathbb{R}^q$:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downdownarrows\downdownarrows & & \downdownarrows\downdownarrows & & \downdownarrows\downdownarrows \\ \cdots & \Rrightarrow & \Gamma^{\times 2}\times F\times\mathbb{R}^{\times 2} & \Rrightarrow & \Gamma\times F\times\mathbb{R}^{\times 2} & \rightrightarrows & F\times\mathbb{R}^{\times 2} \\ & & \downdownarrows\downarrow & & \downdownarrows\downarrow & & \downdownarrows\downarrow \\ \cdots & \Rrightarrow & \Gamma^{\times 2}\times F\times\mathbb{R} & \Rrightarrow & \Gamma\times F\times\mathbb{R} & \rightrightarrows & F\times\mathbb{R} \\ & & \downdownarrows & & \downdownarrows & & \downdownarrows \\ \cdots & \Rrightarrow & \Gamma^{\times 2}\times F & \Rrightarrow & \Gamma\times F & \rightrightarrows & F. \end{array} \tag{5.15}$$

(The bisimplicial set is oriented this way to suggest the simplicial version of the transgression diagram (5.1).) We view $\Gamma\backslash F/\mathbb{R}_{\bullet,\bullet}$ as a simplicial resolution of $((\Gamma\backslash F)/\mathbb{R})_\bullet$, and separately of $(\Gamma\backslash(F/\mathbb{R}))_\bullet$, by projecting to the simplicial sets along $p=0$ and $q=0$, respectively. These projections induce pullback maps

$$\begin{array}{ccc} \Omega^1(\Gamma\backslash F/\mathbb{R}_{\bullet,\bullet}) & \xleftarrow[f^*]{} & \Omega^1(((\Gamma\backslash F)/\mathbb{R})_\bullet) \\ {\scriptstyle g^*}\uparrow & & \\ \Omega^1(\Gamma\backslash(F/\mathbb{R})_\bullet). & & \end{array} \tag{5.16}$$

The degree-3 piece of the total complex is

$$\Omega^1(F\times\mathbb{R}^{\times 2}) \oplus \Omega^1(\Gamma\times F\times\mathbb{R}) \oplus \Omega^1(\Gamma^{\times 2}\times F). \tag{5.17}$$

We pick an identification $F \simeq S^1\times\mathbb{R}_+^\times$, such that the right action by $\mathbb{R}$ is given by the exponential map $\mathbb{R}\to\mathbb{R}_+^\times$, followed by multiplication. To describe the differential forms, it is helpful to fix some further notation. We use γ to denote elements of Γ, (θ, v) for elements of F, and x for elements of $\mathbb{R}$. The action maps are given by

$$\begin{aligned} \Gamma \times F &\to F \\ (\gamma, \theta, v) &\mapsto (\gamma(\theta), \gamma'(\theta) \cdot v) \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} F \times \mathbb{R} &\to F \\ (\theta, v, x) &\mapsto (\theta, e^x \cdot v). \end{aligned} \tag{5.19}$$

We are interested in two cocycles.

(1) The starting point is $f^*(x_1 \, \mathrm{d}x_2)$, which in the decomposition (5.17) is

$$z_1 \coloneqq (x_1 \, \mathrm{d}x_2, 0, 0). \tag{5.20}$$

(2) Our goal is to obtain the pullback of the integrand of the Bott-Thurston cocycle (5.10) under g^*. In the decomposition (5.17) this is

$$z_2 \coloneqq (0, 0, \log(\gamma_1' \circ \gamma_2) \, \mathrm{d}(\log(\gamma_2'))). \tag{5.21}$$

We will show these two cocycles are cohomologous in the total complex.

Lemma 5.22. *The cocycles z_1 and z_2 are cohomologous, i.e. their difference is a coboundary:*

$$z_2 - z_1 = (-x_1 \, \mathrm{d}x_2, 0, \log(\gamma_1' \circ \gamma_2) \, \mathrm{d}(\log(\gamma_2'))) = d\beta, \tag{5.23}$$

where $\beta = (-\log(v) \, \mathrm{d}x, \log(\gamma') \, \mathrm{d}\log(v))$ is a degree 2 cocycle in the double complex, with $-\log(v) \, \mathrm{d}x \in \Omega^1(F \times \mathbb{R})$ and $\log(\gamma') \, \mathrm{d}\log(v) \in \Omega^1(\Gamma \times F)$.

See Figure 1 for a visualization of Lemma 5.22 and of (5.23) more specifically.

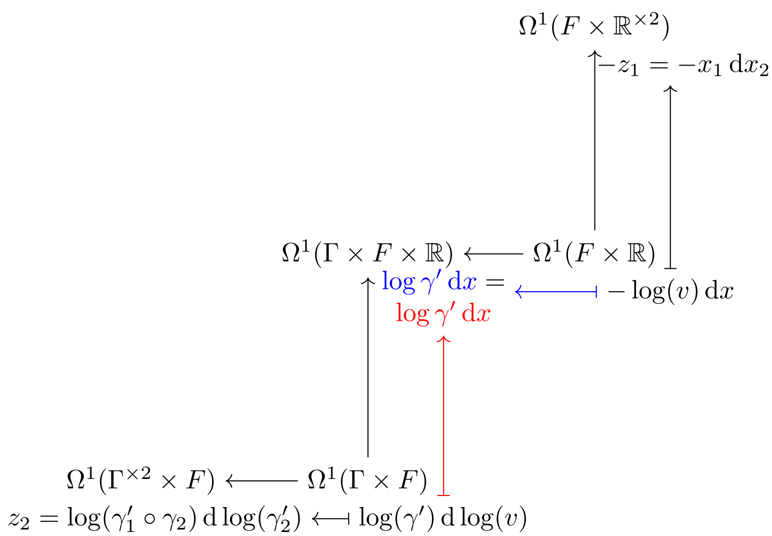

FIGURE 1. A schematic of the proof of Lemma 5.22: that if $\beta = (\log(v) \, \mathrm{d}x, \log(\gamma') \, \mathrm{d}\log(v))$ in total degree 2 in the double complex $\Omega^1(\Gamma \backslash F / \mathbb{R}_{\bullet,\bullet})$, then $\mathrm{d}\beta = z_1 - z_2$.

Proof. Let d^v and d^h be the vertical, resp. horizontal, differentials in the total complex. We need to show the following:

(1) $\mathrm{d}^v(-\log(v)\,\mathrm{d}x) = -x_1\,\mathrm{d}x_2$.
(2) $\mathrm{d}^h(-\log(v)\,\mathrm{d}x) = \log\gamma'\,\mathrm{d}x = \mathrm{d}^v(\log(\gamma')\,\mathrm{d}\log(v))$.
(3) $\mathrm{d}^h(\log(\gamma')\,\mathrm{d}\log(v)) = \log(\gamma_1' \circ \gamma_2)\,\mathrm{d}\log(\gamma_2')$

First we show that

$$\mathrm{d}^v(-\log(v)\,\mathrm{d}x) = -x_1\,\mathrm{d}x_2. \tag{5.24}$$

Recall that we have three maps $\mathrm{d}_0^v, \mathrm{d}_1^v, \mathrm{d}_2^v : F \times \mathbb{R}^2 \to F \times \mathbb{R}$, and $\mathrm{d}^v = \sum(-1)^i(\mathrm{d}_i^v)^*$:

$$\begin{tikzcd} & (\theta, e^{x_1}v, x_2) \\ (\theta, v, x_1, x_2) \arrow[ur, mapsto, "\mathrm{d}_0^v"] \arrow[r, mapsto, "\mathrm{d}_1^v"] \arrow[dr, mapsto, "\mathrm{d}_2^v"] & (\theta, v, x_1 + x_2) \\ & (\theta, v, x_1). \end{tikzcd} \tag{5.25}$$

Thus

$$\begin{aligned} \mathrm{d}^v(-\log(v)\,\mathrm{d}x) &= -\log(e^{x_1}v)\,\mathrm{d}x_2 + \log(v)\,\mathrm{d}(x_1 + x_2) - \log(v)\,\mathrm{d}x_1 \\ &= -x_1\,\mathrm{d}x_2. \end{aligned} \tag{5.26}$$

Similarly, $\mathrm{d}^h = \sum_i(-1)^i(\mathrm{d}_i^h)^*$, and we want to show that $\mathrm{d}^h(-\log(v)\,\mathrm{d}x) = \log\gamma'\,\mathrm{d}x$. In this case there are two maps $\mathrm{d}_0^h, \mathrm{d}_1^h \colon \Gamma \times F \times \mathbb{R} \to F \times \mathbb{R}$:

$$\begin{tikzcd} & (\theta, v, x) \\ (\gamma, \theta, v, x) \arrow[ur, mapsto, "\mathrm{d}_0^h"] \arrow[dr, mapsto, "\mathrm{d}_1^h"'] & \\ & (\gamma(\theta), \gamma'(\theta)v, x). \end{tikzcd} \tag{5.27}$$

Thus

$$\begin{aligned} \mathrm{d}^h(-\log(v)\,\mathrm{d}x) &= -\log(v)\,\mathrm{d}x + \log(\gamma' \cdot v)\,\mathrm{d}x \\ &= \log(\gamma')\,\mathrm{d}x. \end{aligned} \tag{5.28}$$

Next we show that $\mathrm{d}^v(\log(\gamma')\,\mathrm{d}\log(v)) = \log(\gamma')\,\mathrm{d}x$. Recall $\mathrm{d}_0^v, \mathrm{d}_1^v \colon \Gamma \times F \times \mathbb{R} \to \Gamma \times F$ are given by

$$\begin{tikzcd} & (\gamma, \theta, e^x v) \\ (\gamma, \theta, v, x) \arrow[ur, mapsto, "\mathrm{d}_0^v"] \arrow[dr, mapsto, "\mathrm{d}_1^v"'] & \\ & (\gamma, \theta, v), \end{tikzcd} \tag{5.29}$$

and $\mathrm{d}^v = (\mathrm{d}_0^v)^* - (\mathrm{d}_1^v)^*$, so

$$\begin{aligned} \mathrm{d}^v(\log(\gamma')\,\mathrm{d}\log(v)) &= \log(\gamma')\,\mathrm{d}\log(e^x \cdot v) - \log(\gamma')\,\mathrm{d}\log(v) \\ &= \log(\gamma')\,\mathrm{d}x. \end{aligned} \tag{5.30}$$

Lastly we need to show that

$$\mathrm{d}^h(\log(\gamma')\,\mathrm{d}\log(v)) = \log(\gamma_1' \circ \gamma_2)\,\mathrm{d}\log(\gamma_2'). \tag{5.31}$$

We have three maps $\mathrm{d}_0^h, \mathrm{d}_1^h, \mathrm{d}_2^h : \Gamma^2 \times F \to \Gamma \times F$, given by

$$\begin{tikzcd}[row sep=small] & (\gamma_2, \theta, v) \\ (\gamma_1, \gamma_2, \theta, v) \arrow[ur, mapsto, "\mathrm{d}_0^h"] \arrow[r, mapsto, "\mathrm{d}_1^h"] \arrow[dr, mapsto, "\mathrm{d}_2^h"'] & (\gamma_1 \circ \gamma_2, \theta, v) \\ & (\gamma_1, \gamma_2(\theta), \gamma_2'(\theta) v), \end{tikzcd} \tag{5.32}$$

and $\mathrm{d}^h = (\mathrm{d}_0^h)^* - (\mathrm{d}_1^h)^* + (\mathrm{d}_2^h)^*$:

$$(\mathrm{d}_0^h)^*(\log(\gamma') \,\mathrm{d}\log(v)) = \log \gamma_2'(\theta) \,\mathrm{d}\log v. \tag{5.33a}$$

$$\begin{aligned}(\mathrm{d}_1^h)^*(\log(\gamma') \,\mathrm{d}\log(v)) &= \log(\gamma_1 \circ \gamma_2)'(\theta) \,\mathrm{d}\log v \\ &= \log(\gamma_1'(\gamma_2(\theta))) \,\mathrm{d}\log v + \log \gamma_2'(\theta) \,\mathrm{d}\log v.\end{aligned} \tag{5.33b}$$

$$\begin{aligned}(\mathrm{d}_2^h)^*(\log(\gamma') \,\mathrm{d}\log(v)) &= \log \gamma_1'(\gamma_2(\theta)) \,\mathrm{d}(\log(\gamma_2'(\theta) v)) \\ &= \log \gamma_1'(\gamma_2(\theta)) \,\mathrm{d}\log\ (\gamma_2'(\theta)) + \log \gamma_1'(\gamma_2(\theta)) \,\mathrm{d}\log v.\end{aligned} \tag{5.33c}$$

Thus

$$\begin{aligned}\mathrm{d}^h(\log(\gamma') \,\mathrm{d}\log(v)) &= ((\mathrm{d}_0^h)^* - (\mathrm{d}_1^h)^* + (\mathrm{d}_2^h)^*)(\log(\gamma') \,\mathrm{d}\log(v)) \\ &= \log(\gamma_1' \circ \gamma_2) \,\mathrm{d}\log(\gamma_2').\end{aligned} \tag{5.34}$$

This completes the proof of Lemma 5.22. □

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Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907
Email address: adebray@purdue.edu

Department of Mathematics, Harvard University, Science Center Room 325, 1 Oxford Street, Cambridge, MA 02138
Email address: yuleonliu@math.harvard.edu

Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter (550), Woodstock Road, Oxford OX2 6GG
Email address: weis@maths.ox.ac.uk