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RESEARCH ARTICLE

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# Stable diffeomorphism classification of some unorientable 4-manifolds

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**Abstract**

Kreck's modified surgery theory reduces the classification of closed, connected 4-manifolds, up to connect sum with some number of copies of $S^2 \times S^2$, to a series of bordism questions. We implement this in the case of unorientable 4-manifolds $M$ and show that for some choices of fundamental groups, the computations simplify considerably. We use this to solve some cases in which $\pi_1(M)$ is finite of order 2 mod 4: under an assumption on cohomology, there are nine stable diffeomorphism classes for which $M$ is pin$^+$, one stable diffeomorphism class for which $M$ is pin$^-$, and four stable diffeomorphism classes for which $M$ is neither. We also determine the corresponding stable homeomorphism classes.

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## 1 | INTRODUCTION

The classification of closed 4-manifolds up to diffeomorphism is impossible in general: a solution would also solve the word problem for groups. Even if one fixes the fundamental group to avoid this problem, the classification is still currently intractable. For this reason, topologists study weaker classifications of 4-manifolds which are coarse enough to be calculable yet fine enough to be useful.

Stable diffeomorphism is an example of such an invariant. Two closed 4-manifolds $M$ and $N$ are *stably diffeomorphic* if there are $m, n \geqslant 0$ such that $M \# m(S^2 \times S^2)$ is diffeomorphic to

$N \# n(S^2 \times S^2)$. This notion of equivalence has applications to quantum topology: for example, Reutter [26, Theorem A] shows that the partition functions of 4d semisimple-oriented TFTs are insensitive to stable diffeomorphism along the way to showing that such TFTs cannot distinguish homotopy-equivalent closed, oriented 4-manifolds. And stable diffeomorphism classes are computable: once the fundamental group $G$ is fixed, Kreck [21] shows how to reduce the classification of 4-manifolds up to stable diffeomorphism to a collection of bordism computations, and for many choices of $G$, the classification of closed, connected, oriented 4-manifolds with $\pi_1(M) \cong G$ up to stable diffeomorphism has been completely worked out, thanks to work of Wall [33], Teichner [30], Spaggiari [28], Crowley-Sixt [6], Politarczyk [25], Kasprowski–Land–Powell–Teichner [16], Pedrotti [23], Hambleton–Hildum [13], and Kasprowski–Powell–Teichner [17].

Researchers interested in topological manifolds also study *stable homeomorphism* of topological manifolds, that is, homeomorphism after connect-summing with some number of copies of $S^2 \times S^2$. Kreck's theorem applies to this case too, reframing the question in terms of bordism of topological manifolds. Stable homeomorphism classifications are studied by Teichner [30, §5], Wang [35], Hambleton–Kreck–Teichner [14], Kasprowski–Land–Powell–Teichner [16, §§4–5], Hambleton–Hildum [13], and Kasprowski–Powell–Teichner [17, §2.3],

Much less work has been done on unorientable 4-manifolds, even though the theory still works and is simpler in some cases, as we explain below. There is some work in the literature, such as that of Kreck [20], Wang [35], Kurazono [22], Davis [7], and Friedl–Nagel–Orson–Powell [10, §12].

The goal of this paper is to compute sets of stable diffeomorphism and stable homeomorphism classes for a class of unorientable 4-manifolds, as well as determining the corresponding complete stable diffeomorphism and homeomorphism invariants. As a consequence of our Theorem 2.1, for many finite groups $G$, the classification of stable diffeomorphism or homeomorphism classes of unorientable 4-manifolds with $\pi_1(M) \cong G$ reduces to the stable classifications for a smaller 2-group. For example, we show that the stable diffeomorphism, respectively, homeomorphism classification when $\pi_1(M) \cong \mathbb{Z}/2$ determines the stable diffeomorphism, respectively, homeomorphism classification for some groups $G$ of order 2 mod 4. We then compute these classifications using Kreck's techniques.

Suppose that $G$ is the fundamental group of an unorientable manifold. Then there is an extension

$$1 \longrightarrow K \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow 1, \tag{1.1}$$

where $G \twoheadrightarrow \mathbb{Z}/2$ is defined by classifying loops as orientation-preserving or orientation-reversing. Therefore $\mathbb{Z}/2$ acts on $K$.

**Theorem** (Main theorem). *Let $G$ be a finite group of order* 2 mod 4,[†] *and suppose that in (1.1), $\mathbb{Z}/2$ acts trivially on $H^*(BK)$.*

(1) *There are fourteen equivalence classes of closed, connected, unorientable 4-manifolds $M$ up to stable diffeomorphism: nine for which $M$ is pin$^+$, one for which $M$ is pin$^-$, and four for which $M$ is neither.*
(2) *There are twenty equivalence classes of closed, connected, unorientable topological 4-manifolds $M$ up to stable homeomorphism: ten for which $M$ is pin$^+$, two for which $M$ is pin$^-$, and eight for which $M$ is neither.*

[†] Equivalently, the Sylow 2-subgroup of $G$ is isomorphic to $\mathbb{Z}/2$.

This is a combination of Theorems 3.1, 3.5, 4.2, and 4.5. In those theorems we also determine complete stable diffeomorphism/homeomorphism invariants for these manifolds. The classification for $M$ neither pin$^+$or pin$^-$can be extracted from work of Davis [7, Theorem 2.3], but the other parts are new.

We prove these theorems by establishing isomorphisms of bordism groups. Specifically, Kreck's modified surgery theory associates to $G$ a set of symmetry types $\xi : B \to BO$ and expresses the set of stable diffeomorphism classes in terms of the bordism groups $\Omega_4^\xi$; we show that when $|G| \equiv 2 \bmod 4$ and the assumption about $H^*(K)$ holds, the Thom spectra of these symmetry types are homotopy equivalent to the Thom spectra for unoriented, pin$^+$, and pin$^-$bordism. In the smooth case, the bordism groups $\Omega_4^{\mathrm{O}}$, $\Omega_4^{\mathrm{Pin}^+}$, and $\Omega_4^{\mathrm{Pin}^-}$ are well known. The topological versions of these bordism groups are less well known, but Kirby–Taylor [19, §9] compute $\Omega_4^{\mathrm{TopPin}^\pm}$ and provide enough information for us to compute $\Omega_4^{\mathrm{Top}}$, which we do in Proposition 4.7.

The argument we use to establish the isomorphism from $\xi$-bordism to a simpler kind of bordism applies to more general choices of $\pi_1(M)$.

**Theorem 2.1.** *Suppose that G is a finite group fitting into an extension*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} P \longrightarrow 1, \tag{1.2}$$

*where $|K|$ is odd and $P$ is a 2-group, and suppose that $P$ acts trivially on $H^*(BK)$. For any unorientable virtual vector bundle $V \to BP$, $\varphi$ induces an equivalence of Thom spectra $(BG)^{\varphi^* V} \xrightarrow{\simeq} (BP)^V$.*

The Pontrjagin–Thom construction turns this equivalence into isomorphisms of bordism groups from the unorientable symmetry types Kreck associates to $G$ to the unorientable symmetry types for $P$, which we can use to compute stable diffeomorphism classes. The proof strongly requires the assumption that $V$ is unorientable; nothing like this is true in the oriented case.

Our main theorem above covers the case $|G| \equiv 2 \bmod 4$. The next step would be to consider $P \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/4$, which would suffice for many groups $G$ of order 4 mod 8. For these choices of $P$, many of the needed bordism groups have already been computed in the literature for other applications. For $P \cong \mathbb{Z}/4$, see Botvinnik–Gilkey [3, §5]; for $P \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, see work of Guo–Ohmori–Putrov–Wan–Wang [12, §7], the author in [15, Appendix F] and [8, §4.4], and Wan–Wang–Zheng [34, Appendix A].

We begin in §1 with a quick review of Kreck's theorem [21] on stable diffeomorphism classes of 4-manifolds within a given 1-type. In §2, we study the Thom spectra of unorientable vector bundles over $BG$, where $G$ is a finite group, proving Theorem 2.1. In §3, we specialize to the case where $|G| \equiv 2 \bmod 4$, determining the three possible normal 1-types and computing the sets of stable diffeomorphism classes for them. We prove Theorems 3.1 and 3.5, which together form the smooth part of the main theorem above. In Example 3.4, we discuss an example: $\mathbb{RP}^4$ is homeomorphic but not stably diffeomorphic to Cappell–Shaneson's fake $\mathbb{RP}^4$. This fact was known to Cappell–Shaneson [4, 5] and the proof using Kreck's surgery theory is due to Stolz [29]. In §4, we consider stable homeomorphism classes of topological manifolds with $|\pi_1(M)| \equiv 2 \bmod 4$, and prove Theorems 4.2 and 4.5, which form the topological part of the main theorem above.

## 2 | REVIEW: NORMAL 1-TYPES, NORMAL 1-SMOOTHINGS, AND STABLE DIFFEOMORPHISM CLASSES

We review some standard definitions in this area. We will always assume that our manifolds are closed and connected. Except in §4, we also assume that they are smooth.

**Definition 2.1.** A *normal 1-type* of a manifold $M$ is a fibration $\xi : B \to B\mathrm{O}$ such that there is a lift of the map $\nu : M \to BO$ classifying the stable normal bundle of $M$ to a map $\widetilde{\nu} : M \to B$ such that $\xi \circ \widetilde{\nu} = \nu$, $\widetilde{\nu}$ is 2-connected, and $\xi$ is 2-coconnected.

A choice of such a lift is called a *normal 1-smoothing* of $M$.

Any two normal 1-types of a given manifold are homotopy equivalent as spaces over $B\mathrm{O}$, so we will abuse notation and say "the" normal 1-type.

The map $\xi : B \to B\mathrm{O}$ determines a bordism theory of manifolds with a lift of the stable normal bundle across $\xi$, which we denote $\Omega_*^\xi$; a normal 1-smoothing of $M$ determines a class in this bordism group. Different normal smoothings of the same manifold do not always define the same class in $\Omega_*^\xi$.

Let $V_{\mathrm{SO}} \to B\mathrm{SO}$, $V_{\mathrm{Spin}} \to B\mathrm{Spin}$, and so on, denote the tautological stable vector bundles over their respective spaces. We use the convention that maps to $B\mathrm{O}$ are represented by rank-zero virtual vector bundles, which is why we write $E - \dim E$ in (2.3), for example.

**Example 2.2** Kreck [21, §2, Proposition 2]. When $M$ is unorientable, Kreck classifies the possible normal 1-types of $M$ into two families: almost spin and totally non-spin. Let $M' \to M$ be the universal cover of $M$, which is classified by a map $\theta : M \to B\pi_1(M)$.

**Almost spin:** If $M'$ admits a spin structure, $M$ is called *almost spin*. In this case, $w_1(M) = \theta^* x_1$ and $w_2(M) = \theta^* x_2$ for some $x_1, x_2 \in H^*(BG; \mathbb{Z}/2)$. Assume that there is a vector bundle $E \to BG$ such that $w_i(E) = x_i$ for $i = 1, 2$.[†] Then, the normal 1-type of $M$ is

$$\begin{array}{ccc} & & B\mathrm{Spin} \times B\pi_1(M) \\ & \nearrow & \downarrow{\scriptstyle V_{\mathrm{Spin}} \oplus (E - \dim E)} \\ M & \xrightarrow{\nu} & B\mathrm{O}. \end{array} \tag{2.3}$$

**Totally non-spin:** If $M'$ does not admit a spin structure, $M$ is called *totally non-spin*. In this case, $w_1(M) = \theta^* x$ for some $x \in H^1(BG; \mathbb{Z}/2)$. Let $E \to BG$ be a line bundle with $w_1(E) = x$. Then the normal 1-type of $M$ is

$$\begin{array}{ccc} & & B\mathrm{SO} \times B\pi_1(M) \\ & \nearrow & \downarrow{\scriptstyle V_{\mathrm{SO}} \oplus (E - 1)} \\ M & \xrightarrow{\nu} & B\mathrm{O}. \end{array} \tag{2.4}$$

[†] The assumption that $x_1$ and $x_2$ are Stiefel–Whitney classes of some vector bundle on $BG$ is not true for arbitrary $G$, but is true for the groups we consider in this paper. The classification of normal 1-types into almost spin versus totally non-spin holds for all $G$, however.

Because $S^2 \times S^2$ has trivial stable normal bundle, taking connect sum with $S^2 \times S^2$ does not change the normal 1-type of a 4-manifold; thus, the classification of 4-manifolds up to stable diffeomorphism can proceed one normal 1-type at a time. Moreover, because $S^2 \times S^2$ is null-bordant, one might conclude that stably diffeomorphic 4-manifolds $M$ and $N$ are bordant — or, more precisely, that $M$ and $N$ admit normal 1-smoothings which are bordant in $\Omega_4^\xi$. So a plausible lower bound for the set of stable diffeomorphism classes with normal 1-type $\xi$ would be $\Omega_4^\xi$ modulo some identifications arising from inequivalent normal 1-smoothings of the same underlying manifold. Remarkably, this turns out to be a complete classification!

**Theorem 2.5** Kreck [21, Theorem C; §3, Proposition 4].

(1) *If $M$ and $N$ are 4-manifolds of the same normal 1-type $\xi : B \to BO$ admitting normal 1-smoothings which are bordant in $\Omega_4^\xi$, then $M$ is stably diffeomorphic to $N$.*

(2) *If $\pi_1(B)$ is finite, every class in $\Omega_4^\xi$ can be realized as the normal 1-smoothing of a 4-manifold with normal 1-type $\xi$.*

*The upshot is that if* $\mathrm{Aut}(\xi)$ *denotes the group of fiber homotopy equivalences of* $\xi : B \to BO$*, the set of stable diffeomorphism classes of 4-manifolds with normal 1-type* $\xi$ *is* $\Omega_4^\xi/\mathrm{Aut}(\xi)$.

The set of bordism classes of normal 1-smoothings of a given 4-manifold is contained within an $\mathrm{Aut}(\xi)$-orbit of $\Omega_4^\xi$, so one effect of the quotient is to identify these as all coming from the same manifold.

This illustrates the standard way to calculate stable diffeomorphism classes: determine $\Omega_4^\xi$ and then determine the $\mathrm{Aut}(\xi)$-action. These bordism groups are the homotopy groups of the Thom spectrum $M\xi$ of $\xi$, so in the next section we begin the calculation of stable diffeomorphism classes by simplifying $M\xi$.

## 3 | SIMPLIFYING THOM SPECTRA

Theorem 2.5 tells us to investigate the Thom spectra of the normal 1-types in Example 2.2. In both cases, the vector bundle is an exterior direct sum, so the Thom spectra split, as $MSpin \wedge (B\pi_1(M))^V$ in the almost spin case and $MSO \wedge (BG)^V$ in the totally non-spin case, where $V$ is a rank-zero unoriented virtual vector bundle. We attack the problem by simplifying $(B\pi_1(M))^V$ for some choices of $\pi_1(M)$.

**Theorem 3.1.** *Suppose that $G$ is a finite group fitting into an extension*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} P \longrightarrow 1, \tag{3.2}$$

*where $|K|$ is odd and $P$ is a 2-group, and suppose that $P$ acts trivially on $H^*(BK)$. For any unorientable virtual vector bundle $V \to BP$, $\varphi$ induces an equivalence of Thom spectra $(BG)^{\varphi^*V} \xrightarrow{\simeq} (BP)^V$.*

We will prove this in a series of lemmas.

**Definition 3.3.** Let $H$ be a group, $A$ be an abelian group, and $\alpha \in H^1(BH;\mathbb{Z}/2)$. Using the identification $H^1(BH;\mathbb{Z}/2) \cong \mathrm{Hom}(H,\mathbb{Z}/2)$, let $A_\alpha$ be the $\mathbb{Z}[H]$-module which is the abelian group $\mathbb{Z}$ with the $H$-action in which $g \in H$ acts by $(-1)^{\alpha(g)}$.

**Lemma 3.4.** *In the situation of Theorem 3.1, both $\widetilde{H}^*((BG)^{\varphi^*V})$ and $\widetilde{H}^*((BP)^V)$ are 2-torsion.*

*Proof.* Using Definition 3.3, we define the $\mathbb{Z}[P]$-module $A_{w_1(V)}$ and the $\mathbb{Z}[G]$-module $A_{w_1(\varphi^*P)}$, which is isomorphic to the pullback of $A_{w_1(V)}$ by $\varphi$. The Thom isomorphism provides isomorphisms of graded abelian groups

$$H^*(BP;\mathbb{Z}_{w_1(V)}) \xrightarrow{\cong} \widetilde{H}^*((BP)^V), \tag{3.5a}$$

$$H^*(BG;\mathbb{Z}_{w_1(\varphi^*V)}) \xrightarrow{\cong} \widetilde{H}^*((BG)^{\varphi^*V}), \tag{3.5b}$$

so we will prove the lemma using group cohomology — specifically, using the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(BP;(H^q(BK;\mathbb{Z}))_{w_1(V)}) \Longrightarrow H^{p+q}(BG;\mathbb{Z}_{w_1(\varphi^*V)}). \tag{3.6}$$

Here it is crucial that $P$ acts trivially on $H^*(BK)$; otherwise, we would have a different local coefficient system than $H^q(BK;\mathbb{Z})_{w_1(V)}$ in (3.6).

Since $E_2^{p,q} \cong H^p(BP;M_q)$ for some $\mathbb{Z}[P]$-module $M_q$, $E_2^{p,q}$ is 2-torsion for $p > 1$ by Maschke's theorem.[†] When $p = 0$,

$$E_2^{0,q} \cong H^0(BP;H^q(BK)_{w_1(V)}) \cong (H^q(BK)_{w_1(V)})^P. \tag{3.7}$$

We will show this vanishes. First, $H^q(BK)$ is $\mathbb{Z}$ for $q = 0$ and is odd-primary torsion for $q > 0$ (by Maschke's theorem, because $2 \nmid \#K$). Therefore if $a \in H^q(BK)$ and $-a = a$, $a = 0$. Since $w_1(V) \neq 0$, there is some $g \in P$ which acts on $\mathbb{Z}_{w_1(V)}$ as $-1$, hence also acts on $H^q(BK)_{w_1(V)}$ as $-1$, so the subgroup of invariants of $H^q(BK)_{w_1(V)}$ is $\{0\}$.

Considering the line $q = 0$ proves $H^*(BP;\mathbb{Z}_{w_1(V)})$ is 2-torsion. For $H^*(BG;\mathbb{Z}_{w_1(\varphi^*V)})$, we have shown the $E_2$-page is 2-torsion, so the graded abelian group the spectral sequence converges to is also 2-torsion. □

**Lemma 3.8.** *With $G$ and $P$ as in Theorem 3.1, $\varphi^*\colon H^*(BP;\mathbb{Z}/2) \to H^*(BG;\mathbb{Z}/2)$ is an isomorphism of graded rings.*

*Proof.* Since $K$ has odd order, its mod 2 cohomology is $\mathbb{Z}/2$ in degree 0 and vanishes elsewhere, so the result follows from the Leray–Hirsch theorem applied to the fibration $BK \to BG \to BP$ induced by (3.2). □

[†] We use Maschke's theorem as follows: if $G$ is a finite group and $k$ is a field of characteristic 0 or characteristic $\ell \nmid \#G$, the category of $k[G]$-modules is semisimple. Therefore all positive-degree Ext groups vanish, in particular $H^m(BG;M) \cong \mathrm{Ext}^m_{k[G]}(\mathbb{Z},M)$ for any $k[G]$-module $M$ and $m > 1$. Combined with the universal coefficient theorem, this implies that for any $\mathbb{Z}[G]$-module $M$ and $m > 1$, $H^m(G;M)$ is torsion ($k = \mathbb{Q}$), and lacks $\ell$-torsion if $\ell \nmid \#G$.

*Proof of Theorem* 3.1. Use the homology Whitehead theorem: if $f : X \to Y$ is a map of bounded-below spectra which induces an isomorphism on rational cohomology and on mod $p$ cohomology for every prime $p$, then $f$ is a homotopy equivalence. Lemma 3.4 and the universal coefficient theorem imply that if $k = \mathbb{Q}$ or $k = \mathbb{Z}/p$ for an odd prime $p$, $\widetilde{H}^*((BG)^{\varphi^* V}; k)$ and $\widetilde{H}^*((BP)^V; k)$ both vanish, so the map between them is vacuously an isomorphism. The sole remaining case is $p = 2$. Since $1 \equiv -1 \bmod 2$, $(\mathbb{Z}/2)_{w_1(V)}$ carries the trivial $P$-action; thus, the Thom isomorphism has the form

$$H^*(BP; \mathbb{Z}/2) \xrightarrow{\cong} \widetilde{H}^*((BP)^V; \mathbb{Z}/2). \tag{3.9a}$$

Analogously, there is a Thom isomorphism

$$H^*(BG; \mathbb{Z}/2)) \xrightarrow{\cong} \widetilde{H}^*((BG)^{\varphi^* V}; \mathbb{Z}/2). \tag{3.9b}$$

As the Thom isomorphism is functorial with respect to pullbacks of vector bundles, Lemma 3.8 lifts to imply that

$$\varphi^* : \widetilde{H}^*((BP)^V; \mathbb{Z}/2) \longrightarrow \widetilde{H}^*((BG)^{\varphi^* V}; \mathbb{Z}/2) \tag{3.10}$$

is an isomorphism. □

## 4 | THE CASE $|\pi_1(X)| \equiv 2 \bmod 4$

If $M$ is an unorientable manifold, the description of loops as orientation-preserving or orientation-reversing defines a surjection $p : \pi_1(M) \to \mathbb{Z}/2$, so $\pi_1(M)$ cannot have odd order. Thus the simplest case occurs when $|\pi_1(M)| \equiv 2 \bmod 4$, so that $|\ker(p)|$ is odd; equivalently, the Sylow 2-subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}/2$. For the rest of this section, fix such a group $G$, and assume that $\mathbb{Z}/2$ acts trivially on $H^*(B\ker(p))$.

In this case, Theorem 3.1 applies to show that if $\mathbb{Z}/2$ acts trivially on $H^*(B\ker(p))$ and $V \to B\mathbb{Z}/2$ is any unorientable virtual vector bundle, the map $(B\pi_1(M))^{p^* V} \xrightarrow{\simeq} (B\mathbb{Z}/2)^V$ is an equivalence.

Let $\sigma \to B\mathbb{Z}/2$ denote the tautological line bundle and $x := w_1(\sigma) \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$, so $H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$. Because $\ker(p)$ has odd order, the Leray–Hirsch theorem implies $p^* : H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \to H^*(B\pi_1(M); \mathbb{Z}/2)$ is an isomorphism.

### 4.1 | The almost spin case

Example 2.2 shows that there are two unorientable normal 1-types in this case: $w_1(\nu) \neq 0$, so it must be the pullback of $p^* x \in H^1(B\pi_1(M); \mathbb{Z}/2)$, and for $w_2$, we have two choices: $w_2 = 0$ (the normal bundle is pin$^+$) and $w_2 = p^* x^2$ (the normal bundle is pin$^-$).

Recall that for a manifold $M$, $M$ is pin$^\pm$ (that is, the tangent bundle is pin$^\pm$) if and only if the normal bundle is pin$^\mp$. A (tangential) pin$^+$4-manifold $M$ has a $\mathbb{Z}/16$-valued invariant given by the $\eta$-invariant of a twisted Dirac operator [29, §4]; let $\eta'$ be the invariant assigning to a pin$^+$4-manifold $M$ the image of this $\eta$-invariant in the nine-element set $(\mathbb{Z}/16)/(x \sim -x)$. We will see in the proof of Theorem 3.1 that all pin$^+$structures on $M$ give the same value of $\eta'$, so we may define it as an invariant of manifolds which admit a pin$^+$structure, without choosing such a structure.

**Theorem 4.1.** *There are nine stable diffeomorphism classes of unorientable 4-manifolds with $\pi_1(M) \cong G$ that admit a (tangential) pin$^+$structure, and there is a single stable diffeomorphism class of manifolds with $\pi_1(M) \cong G$ that admit a (tangential) pin$^-$structure. In the pin$^+$case, $\eta'$ is a complete stable diffeomorphism invariant.*

*Proof.* Both choices of $(w_1, w_2)$ arise from vector bundles: $(p^*x, 0)$ from $p^*\sigma$, and $(p^*x, p^*x^2)$ from $p^*(3\sigma)$. Thus the normal 1-types are

$$V_{\mathrm{Spin}} \oplus (p^*\sigma - 1) :\ B\mathrm{Spin} \times B\pi_1(M) \longrightarrow BO, \tag{4.2a}$$

$$V_{\mathrm{Spin}} \oplus (p^*(3\sigma) - 3) :\ B\mathrm{Spin} \times B\pi_1(M) \longrightarrow BO, \tag{4.2b}$$

and their Thom spectra are $MSpin \wedge (B\pi_1(M))^{p^*\sigma-1}$, respectively, $MSpin \wedge (B\pi_1(M))^{p^*(3\sigma)-3}$. By Theorem 2.1, these are equivalent to $MSpin \wedge (B\mathbb{Z}/2)^{\sigma-1}$, resp. $MSpin \wedge (B\mathbb{Z}/2)^{3\sigma-3}$.

**Theorem 4.3** Peterson [24, §7], Kirby–Taylor [18, Lemma 6]. *There are equivalences $MSpin \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTPin^-$ and $MSpin \wedge (B\mathbb{Z}/2)^{3\sigma-3} \simeq MTPin^+$.*†

These bordism groups are known.

- In the case $w_2(\nu) = 0$, $\Omega_4^\xi \cong \Omega_4^{\mathrm{Pin}^-} \cong 0$ [1, 19] — all 4-manifolds with this normal 1-type are stably diffeomorphic.
- When $w_2(\nu) = p^*x^2$, $\Omega_4^\xi \cong \Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}/16$ [11, 18, 19].

Kreck [20, §5] shows that in the pin$^+$case, $\mathrm{Aut}(\xi) \cong \mathbb{Z}/2$ and the action of the non-trivial automorphism on $\mathbb{Z}/16$ sends $x \mapsto -x$. We thus obtain nine equivalence classes: $0, \pm 1, \pm 2, \dots, \pm 7, 8$, detected by the image of the $\eta$-invariant in $(\mathbb{Z}/16)/(x \sim -x)$. □

As a consequence of Kreck's classification in Example 1.2, we have seen that all unorientable, almost spin 4-manifolds $M$ with $\pi_1(M) \cong G$ are either pin$^+$or pin$^-$, and that this determines their normal 1-type. This is not true for more general $G$.

**Example 4.4.** Cappell–Shaneson [4, 5] construct a closed, smooth 4-manifold $Q$ that is homeomorphic but not diffeomorphic to $\mathbb{RP}^4$, and show that $Q$ and $\mathbb{RP}^4$ are not stably diffeomorphic. Stolz [29] gives another proof of this fact by computing the classes of $\mathbb{RP}^4$ and $Q$ in $\Omega_4^\xi/\mathrm{Aut}(\xi)$. We briefly summarize Stolz' proof.

Since $\pi_1(\mathbb{RP}^4) \cong \mathbb{Z}/2$ and $w_2(\mathbb{RP}^4) = 0$, the proof of Theorem 4.1 shows $M\xi \simeq MTPin^+$, $\Omega_4^\xi \cong \mathbb{Z}/16$, and the set of stable diffeomorphism classes is $\Omega_4^\xi/\mathrm{Aut}(\xi) \cong (\mathbb{Z}/16)/(x \sim -x)$. Stolz [29] chooses an isomorphism $\Omega_4^\xi \xrightarrow{\cong} \mathbb{Z}/16$ and shows that it sends the two pin$^+$structures on $\mathbb{RP}^4$ to $\pm 1$ and the two pin$^+$structures on $Q$ to $\pm 9$; therefore, $\mathbb{RP}^4$ and $Q$ are not stably diffeomorphic.

† There is an important subtlety in the names of these spectra in the literature: $MPin^\pm$ denotes the Thom spectra classifying pin$^\pm$ structures on the stable normal bundle, and $MTPin^\pm$ denotes the Thom spectra classifying pin$^\pm$ structures on the stable tangent bundle. There are equivalences $MPin^\pm \simeq MTPin^\mp$. Information on pin$^\pm$ bordism is usually stated in terms of $MTPin^\pm$.

## 4.2 | The totally non-spin case

**Theorem 4.5.** *There are four stable diffeomorphism classes of unorientable, totally non-spin 4-manifolds with $\pi_1(M) \cong G$. The Stiefel–Whitney numbers $w_4$ and $w_2^2$ detect these classes.*

This theorem can also be extracted from work of Davis [7, Theorem 2.3], who computes a different set of invariants.

*Proof.* Example 1.2 shows that there is only one unorientable normal 1-type in this case: $w_1(\nu) \neq 0$, so it must be pulled back from $p^*x \in H^1(B\pi_1(M); \mathbb{Z}/2)$. Since $p^*x = w_1(p^*\sigma)$, the normal 1-type is

$$V_{\mathrm{SO}} \oplus (p^*\sigma - 1) \colon B\mathrm{SO} \times B\pi_1(M) \longrightarrow B\mathrm{O}, \tag{4.6}$$

and its Thom spectrum is $MSO \wedge (B\pi_1(M))^{p^*\sigma-1}$, which by Theorem 2.1 is equivalent to $MSO \wedge (B\mathbb{Z}/2)^{\sigma-1}$.

**Lemma 4.7** Atiyah [2, Proposition 2.3]. *There is an equivalence $MSO \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MO$.*

So $\Omega_4^{\xi} \cong \Omega_4^{\mathrm{O}}$, and $\Omega_4^{\mathrm{O}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ [32, Corollaire following Théorème IV.12]. The $\mathrm{Aut}(\xi)$-action is trivial. To see this, first observe that $\mathrm{Aut}(\mathrm{id} \colon B\mathrm{O} \to B\mathrm{O})$ is trivial, hence acts trivially on $\Omega_4^{\mathrm{O}}$. Thus the $\mathrm{Aut}(\xi)$-orbit of a class in $\Omega_4^{\xi}$ maps to a single class in $\Omega_4^{\mathrm{O}}$, so $\mathrm{Aut}(\xi)$-orbits are singletons. Therefore any complete bordism invariant for $\Omega_4^{\mathrm{O}}$ is also a complete stable diffeomorphism invariant for the normal 1-type $\xi$, such as $(w_2^2, w_4)$. □

*Remark* 3.8. If $M$ is pin$^+$or pin$^-$, then its double cover is spin, and hence $M$ is almost spin. So totally non-spin manifolds are neither pin$^+$nor pin$^-$. Therefore the three normal 1-types that occur when $\pi_1(M) \cong G$ and $M$ is unorientable are the cases pin$^+$, pin$^-$, and neither pin$^+$nor pin$^-$.

## 5 | STABLE HOMEOMORPHISM CLASSES

In order to classify stable homeomorphism classes of topological 4-manifolds, we run the same story, replacing $B\mathrm{O}$ with $B\mathrm{Top}$, where $\mathrm{Top}_n$ is the topological group of homeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ that fix the origin and $\mathrm{Top} := \varinjlim_n \mathrm{Top}_n$. As in the previous section, fix a group $G$ finite of order 2 mod 4 with a surjective map $p \colon G \to \mathbb{Z}/2$, and assume that $\mathbb{Z}/2$ acts trivially on $H^*(B\ker(p))$.

Given a topological manifold $M$, there is a map $\nu \colon M \to B\mathrm{Top}$ called the *stable topological normal bundle*, so we can define normal 1-types, and Kreck's classification argument still applies in the topological setting, this time determining stable homeomorphism classes.

**Lemma 5.1.** *Let $M$ be a closed, unorientable 4-manifold. The possible normal 1-types of $M$ are the same as in Example 1.2, except replacing $B\mathrm{O}$ with $B\mathrm{Top}$, $B\mathrm{SO}$ with $B\mathrm{STop}$, and $B\mathrm{Spin}$ with $B\mathrm{TopSpin}$.*

*Proof.* The proof is very similar to Kasprowski–Land–Powell–Teichner's determination of the possible normal 1-types of topological 4-manifolds in the orientable case [16, Proposition 4.1]. Since the Stiefel–Whitney classes of a manifold are homotopy invariants, notions of almost spin and totally non-spin make sense for topological manifolds. In the almost-spin case, we have to check that a lift $M \to B\mathrm{TopSpin} \times B\pi_1(M)$ is 2-connected: the proof is the same as in the smooth case, because $\pi_2(B\mathrm{TopSpin}) = 0$. For the totally non-spin case, $\pi_2(B\mathrm{STop}) \cong \mathbb{Z}/2$, detected by $w_2$, and since $M$ is totally non-spin, $w_2(M) \neq 0$, so the lift is surjective on $\pi_2$ just as in the smooth case. □

Our arguments below make use of the fact that bordism groups of topological manifolds are homotopy groups of Thom spectra, which requires a transversality argument. In dimension 4, Scharlemann [27] proves the topological transversality theorem that we need. See Teichner [31, §IV] for more information.

Let $E_8$ denote Freedman's $E_8$ manifold [9]. The obstruction to admitting a triangulation defines a bordism invariant $\Omega_4^{\mathrm{Top}} \to \mathbb{Z}/2$ [19, §9] which is non-zero on $E_8$.

## 5.1 | The almost spin case

There are topological versions of spin and pin$^\pm$ structures; see Kirby–Taylor [19, §9] for details. Kirby–Taylor also produce a homomorphism $S : \Omega_4^{\mathrm{TopPin}^+} \to \Omega_2^{\mathrm{TopPin}^-} \cong \Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ sending a pin$^+$topological 4-manifold $M$ to the pin$^-$bordism class of a continuously embedded representative of the Poincaré dual of $w_1(M)^2$, which has an induced pin$^-$structure and a unique smooth structure. Let $S'$ be the invariant sending a topological pin$^+$4-manifold $M$ to the image of $S(M)$ in the set $(\mathbb{Z}/8)/(x \sim -x)$.

**Theorem 5.2.**

(1) *There are ten stable homeomorphism classes of unorientable pin$^+$topological 4-manifolds with $\pi_1(M) \cong G$. These classes are detected by the invariant $S'$ constructed above and the triangulation obstruction.*
(2) *There are two stable homeomorphism classes of unorientable pin$^-$topological 4-manifolds with $\pi_1(M) \cong G$. These classes are detected by the triangulation obstruction.*

*Proof.* Following the same line of argument as in the proof of Theorem 3.1, the two normal 1-types' Thom spectra are $MTopSpin \wedge (B\pi_1(M))^{p^*\sigma-1}$ and $MTopSpin \wedge (B\pi_1(M))^{p^*(3\sigma)-3}$, and Theorem 2.1 simplifies these to $MTopSpin \wedge (B\mathbb{Z}/2)^{\sigma-1}$ and $MTopSpin \wedge (B\mathbb{Z}/2)^{3\sigma-3}$, respectively.

**Lemma 5.3.** *There are equivalences $MTopSpin \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTTopPin^-$ and $MTopSpin \wedge (B\mathbb{Z}/2)^{3\sigma-3} \simeq MTTopPin^+$.*

*Proof.* There are surjective maps $d_n : \mathrm{Top}_n \twoheadrightarrow \{\pm 1\}$ given by assigning to a homeomorphism the automorphism it defines on $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{Z}$. These commute with the inclusions $\mathrm{Top}_n \hookrightarrow \mathrm{Top}_{n+1}$, and passing to the colimit defines a map $d : \mathrm{Top} \twoheadrightarrow \{\pm 1\}$. This is a topological version of assigning an orthogonal matrix its determinant, classifying whether it preserves or reverses orientation. Given a principal Top-bundle $P \to M$, let $\mathrm{Det}(P) \to M$ be the line bundle $P \times_{\mathrm{Top}} \mathbb{R} \to M$, where Top acts on $\mathbb{R}$ through $d$. The maps $\mathrm{Top}_n \times \mathrm{O}_1 \to \mathrm{Top}_n \times \mathrm{Top}_1 \to \mathrm{Top}_{n+1}$ allow us to make sense of "$P \oplus n\,\mathrm{Det}(P)$" as a principal Top-bundle.

We abuse notation for a moment to say that a $G$-structure on a principal Top-bundle $P \to M$ is a reduction of structure group of $P$ from Top to $G$. Then, just as in the smooth case, there is a natural equivalence between the set of $\mathrm{TopPin}^-$-structures on $P$ and the set of TopSpin structures on $P \oplus \mathrm{Det}(P)$, and similarly between the set of $\mathrm{TopPin}^+$-structures on $P$ and the set of TopSpin structures on $P \oplus 3\,\mathrm{Det}(P)$. The proof is the same as in the smooth case. These equivalences are the only facts we need to know about $\mathrm{Pin}^\pm$ in order to prove Theorem 3.3 in the smooth setting, so the argument in the topological setting can proceed in the same way. Therefore by Theorem 2.1, our two normal 1-types are equivalent to $MTTopPin^\pm$. The caveat about switching between $\mathrm{pin}^+$and $\mathrm{pin}^-$when one passes between the tangent and normal bundles still applies here.

**Theorem 5.4** (Kirby–Taylor [19, Theorem 9.2]).

(1) $\Omega_4^{\mathrm{TopPin}^-} \cong \mathbb{Z}/2$, *generated by* $E_8$.
(2) $\Omega_4^{\mathrm{TopPin}^+} \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$, *with* $\mathbb{RP}^4$ *generating the* $\mathbb{Z}/8$ *summand and* $E_8$ *generating the* $\mathbb{Z}/2$ *summand.*
(3) *The map* $\Omega_4^{\mathrm{Pin}^+} \to \Omega_4^{\mathrm{TopPin}^+}$ *is identified with a map* $\mathbb{Z}/16 \to \mathbb{Z}/8 \oplus \mathbb{Z}/2$ *which surjects onto the first factor and does not hit* $E_8$.[†]
(4) *The homomorphism* $S : \Omega_4^{\mathrm{TopPin}^+} \to \Omega_2^{\mathrm{TopPin}^-} \cong \Omega_2^{\mathrm{Pin}-} \cong \mathbb{Z}/8$ *sends* $\mathbb{RP}^4$ *to a generator.*

Since $\mathbb{Z}/2$ is rigid, we conclude that there are two stable homeomorphism classes in the $\mathrm{pin}^-$case, detected by the triangulation obstruction. For the $\mathrm{pin}^+$case, the same line of reasoning in the proof of Theorem 3.1 allows us to reduce to the case when $\xi$ is a topological $\mathrm{pin}^+$structure, so we can compute the action of $\mathrm{Aut}(\xi)$ on the generators. Since $E_8$ is simply connected, it admits a unique topological $\mathrm{pin}^+$structure, so is fixed by $\mathrm{Aut}(\xi)$. Every topological $\mathrm{pin}^+$structure on $\mathbb{RP}^4$ arises from a smooth $\mathrm{pin}^+$structure, so we can reuse the argument from Theorem 3.1 to conclude that the $\mathrm{Aut}(\xi)$-orbit of $\mathbb{RP}^4$ is again $\pm[\mathbb{RP}^4]$. Therefore the set of stable diffeomorphism classes is $((\mathbb{Z}/8)/(x \sim -x)) \times \mathbb{Z}/2$, which has ten elements, and the triangulation obstruction and $S'$ are together a complete invariant. □

## 5.2 | The totally non-spin case

By Lemma 5.1, there is only one normal 1-type to worry about.

**Theorem 5.5.** *There are eight stable homeomorphism classes of unorientable, totally non-spin topological 4-manifolds with* $\pi_1(M) \cong G$. *The triangulation obstruction and the Stiefel–Whitney numbers* $w_4$ *and* $w_2^2$ *are together a complete stable homeomorphism invariant.*

Again, this can be extracted from a theorem of Davis [7, Theorem 2.3], who uses a different but equivalent set of invariants.

*Proof.* Following the same line of reasoning as in Theorem 3.5, Lemma 4.1 tells us we only have one normal 1-type, and its Thom spectrum is $MSTop \wedge (B\mathbb{Z}/2)^{\sigma-1}$.

**Lemma 5.6.** *There is an equivalence* $MTop \simeq MSTop \wedge (B\mathbb{Z}/2)^{\sigma-1}$.

[†] The identification of the kernel of $\Omega_4^{\mathrm{Pin}^+} \to \Omega_4^{\mathrm{TopPin}^+}$ with $\mathbb{Z}/2$, generated by the K3 surface, is an earlier theorem of Kreck [20, §5].

*Proof.* The proof goes through as in the smooth case, since we have a determinant map and the fact that for any Top-bundle $P \to M$, $P \oplus \mathrm{Det}(P)$ is canonically oriented, analogously to the smooth case. □

So we need to calculate $\Omega_4^{\mathrm{Top}}$.

**Proposition 5.7.** $\Omega_4^{\mathrm{Top}} \cong (\mathbb{Z}/2)^{\oplus 3}$*, with a basis given by the classes of* $\mathbb{RP}^4$, $\mathbb{RP}^2 \times \mathbb{RP}^2$*, and* $E_8$*. The Stiefel–Whitney numbers* $w_4$ *and* $w_2^2$ *and the triangulation obstruction are linearly independent on this bordism group.*

*Proof.* Draw the Atiyah–Hirzebruch spectral sequence computing $\Omega_4^{\mathrm{Top}}$ as $\Omega_4^{\mathrm{STop}}((B\mathbb{Z}/2)^{\sigma-1})$. It collapses for degree reasons in total degree 4 and below, and the 4-line of the $E_\infty$-page has order 8. Therefore it suffices to find three linearly independent non-zero elements of $\Omega_4^{\mathrm{Top}}$, which can be done by computing $w_4$, $w_2^2$, and the triangulation obstruction on $\mathbb{RP}^4$, $\mathbb{RP}^2 \times \mathbb{RP}^2$, and $E_8$. □

Just as in the smooth case, $\mathrm{Aut}(\xi)$ acts trivially. □

Remark 3.8 also applies in the topological case: the three normal 1-types for unorientable topological manifolds with $\pi_1(M) \cong G$ are precisely the cases where $M$ has a topological pin$^+$structure, $M$ has a topological pin$^-$structure, and $M$ has neither.

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