ON TACOLOGICAL QUANTUM FIELD THEORY

ARUN DEBRAY
APRIL 1, 2021

Contents

1. Introduction

Recently, categorical methods have become popular in the study of food, with work such as Slawvere's sub object classifier and its application to the ham sandwich theorem; J.H.C. Whitebread's theory of mapping scones; and Morehouse's monadic definition of a burrito [Mor15]. In the spirit of turning coffee into theorems, we instead use food-theoretic methods to study a class of objects in math and physics: *tacological quantum field theories* (tacoQFTs).

TacoQFTs lie at the intersection of math and physics, being mathematical objects built to model real-world, physical phenomena, such as flavor conservation in the Standard Model or the study of "on-taco-shell" configurations in more general QFTs. Once only known in Mexico and the American Southwest, tacoQFTs have recently become popular among researchers across the world. TacoQFT uses food-theoretic definitions and theorems in a relatively explicit way; the purpose of this article is to introduce key definitions and examples in the field. Where we whet the reader's appetite for more, we provide references for the reader to dig into.

In §2, we introduce boardism categories and define tacoQFTs. In §3, we give several examples, focusing on Burned-Salmon theory and its flaming anomaly. In §4, we discuss why to ensure full authenticity in a tacoQFT, one must eat fully locally, and how this leads to the use of higher categorical methods. Finally, in §5, we discuss apps.

Bon appetit.

2. Boardism categories and defining tacoQFTs

Recall the symmetric monoidal category $\mathcal{F}ood$ of food [Mor15, §1]; the objects are food, the morphisms are recipes, and the symmetric monoidal structure is disjoint union. To define tacoQFTs, we must modify this category slightly.

Definition 2.1. Fix an n-dimensional cutting board B. The *boardism category* $\mathcal{B}oard_{n,n-1}^B$ is the symmetric monoidal category whose objects are configurations of ingredients on B, and whose morphisms are time-evolutions from one configuration to another. The symmetric monoidal structure is again disjoint union.

Definition 2.2. A *tacological quantum field theory (tacoQFT)* is a symmetric monoidal functor

$$Z\colon \mathcal{B}oard_{n,n-1}^B \longrightarrow \mathcal{F}ood.$$

That is, a tacoQFT is a functorial way of producing food from configurations of ingredients which factors under disjoint unions.

Remark 2.3. For readers approaching tacoQFTs from the perspective of physics or Hegel, this definition may be a bit abstruse, so we pause here to motivate it.

This definition is built on Adams' key insight [Ada80] that "time is an illusion; lunchtime doubly so." This leads us to understand a tacoQFT by decomposing it into time-slices and understanding how it behaves under time evolution. Once we agree to slice time, it is natural to consider cutting boards, and the full definition follows quickly. $\blacktriangleleft$

Different choices of cutting board B give rise to different notions of tacoQFT, and it is important to choose the correct kind for the application you have in mind. Some cutting boards are framed, though it is difficult to work with a framed board, since it's attached to the wall. It is more common to work with boards that are oriented or which can be spun.[1]

Given one tacoQFT Z, it is often possible to produce additional tacoQFTs by use of the *tortilla endofunctors* $T\colon \mathcal{F}ood \to \mathcal{F}ood$ [Mor15, §2]. There are many tortilla endofunctors, because there are many kinds of tortillas. Tortilla endofunctors are classified by maps from B to the moduli stack of tortillas.[2]

3. Examples of tacoQFTs

Example 3.1 (Burned-Salmon)**.** Burned-Salmon theory is the preeminent example of a tacoQFT. It is three-dimensional, so we can study it more easily — and in particular, eat it. The Burned-Salmon tacoQFT is delicious (though see below for the flaming anomaly). This tacoQFT was first studied by Witten [Wit89], who used it to define invariants of onion rings, garlic knots, and sausage links, and it has been the target of much followup research.

Fix L a sausage link in space.[3] Burned-Salmon is a *sage theory*, meaning its flavor symmetry is realized with the herb sage. Specifically, fix a level of sage you want the recipe to have, and call it k. Now garnish L with k amount of sage, and compute the path integral

$$\frac{k}{4\text{🥧}} \int_{\text{🌮}} \left(\text{🍖} \wedge \mathrm{d}\text{🍖} + \frac{2}{3}\text{🍖} \wedge \text{🍖} \wedge \text{🍖} \right). \tag{3.2}$$

At this point we must integrate over the space of all sage configurations. This is easier said than done, and indeed it is an open question how to perform this in a mathematically rigorous manner. Some researchers are not too concerned about open issues in tacoQFT, leading to the notion of an *open face tacoQFT*; others, including Reshetikhin-Turaev [RT91], searched for alternate constructions of Burned-Salmon theory. The Reshetikhin-Turaev construction goes through the representation-theoretic notion of *quantum soups*.[4]

Remark 3.3 (The flaming anomaly)*.* Burned-Salmon theory has an important caveat, the *flaming anomaly* — it's simply too spicy to put on an arbitrary configuration of food without additional data.

Take a closer look at (3.2). This expression is not sage-invariant, which prevents the quantum theory from being consistently spiced. Instead, the whole theory is flaming hot! Some researchers are fine with this; others wonder what's the point [Hen17]. Still others modify the board B, such as requiring a *signature structure*, resulting in a signature recipe.

Freed-Teleman show how to make sense of the flaming anomaly in the world of *breaded monoidal categories*, which are monoidal categories satisfying the *challah axioms*. See [Fre12] for more details. $\blacktriangleleft$

Burned-Salmon theory has been studied for a quarter century, but there is still much to learn. $\blacktriangleleft$

Example 3.4 (Steak-sums)**.** Steak-sum theories are a wide class of tacoQFTs which compute the recipe by cutting the food on the board into small pieces; computing the recipe on these pieces, which is simpler; and then assembling the pieces and checking that the result is independent of choices in how we cut. Though several different steak-sum models exist, the best-studied is the TVBW model [TV92, BW96]. Of particular note are the TVBW models for quantum SU_2, the "A_1 steak-sum" models. $\blacktriangleleft$

---

[1] One version of the *spin-statistics theorem* implies that for spun tacoQFTs, we should replace $\mathcal{F}ood$ with a healthier category, the category $s\mathcal{F}ood$ of *superfoods*.

[2] The moduli stack $\underline{\mathrm{Tor}}$ of tortillas has several connected components, and some researchers take the viewpoint that it is best to analyze different components as separate stacks. For example, there are connected components of whole wheat and white tortillas; the former can be made into a coarse moduli space, and the latter cannot. In both cases, though, important properties can be deduced from their germs.

[3] The story for onion rings and garlic knots is essentially the same.

[4] Quantum soups are not soups, but instead a kind of category.

Example 3.5 (Fro-beans algebras)**.** TacoQFTs in dimension 2 are completely classified. This is a "forklore theorem" — it was proven in several different kitchens before it was formally written down. Every 2d tacoQFT is equivalent to a *Fro-beans algebra.* A Fro-beans algebra provides one important piece of data: a perfect pairing of any recipe with some kind of beans. Black beans, pinto beans, has beens, red beans; in a tacoQFT built out of a Fro-beans algebra, we know exactly how to pair them with the recipe. ◂

Example 3.6 (Theory $\mathcal{EGGS}$)**.** Breakfast tacos, aka Theory $\mathcal{EGGS}$, are a new and exciting tacoQFT, predicted by physicists to unify a great deal of existing recipes and constructions. This theory was first constructed by compactifying type IIB string cheese theory, but has since been studied with many different methods. There is some consternation as to where Theory $\mathcal{EGGS}$ was first discovered, whether in Austin or San Antonio, but I can proudly say it was discovered at none other than the University of Texas at Austin: see [LS18].

Theory $\mathcal{EGGS}$ depends on a choice of ADE Lie algebra, and we suggest enjoying it with a nice lemonADE. You could also begin with a simply laced Dynkin diagram; we will not recommend anything simply laced for the reader to enjoy it with. ◂

4. Eat (fully) local: higher categories

The astute reader has no doubt noticed something essential missing from the tacoQFTs we discussed in the previous section — there is no guarantee of authenticity! Any mathematician can put together a few ingredients into a recipe and produce a tacoQFT satisfying Definition 2.2, but the tacoQFTs that arise in nature have considerably more structure: *(full) locality.* Intuition from physics tells us that the authentic tacoQFTs are the best-tasting ones, and that to eat authentic one must eat local. However, it is not immediately clear how to translate this into the mathematical formalism.

When studying local tacoQFTs, one of the first things to notice is the high number of categories they can belong to. Breakfast tacoQFTs. Street tacoQFTs.[5] TacoQFTs al pastor. And many more categories. The sheer profusion of categories of tacoQFTs led researchers to bring in the theory of *higher categories* to model fully local tacos. The suggestion of "taco trucks on every corner" leads us to consider cutting boards with corners, and more generally higher-dimensional cutting boards with higher corners, giving rise to symmetric monoidal boardism n-categories.

Specifically, consider an n-dimensional cutting board B with corners of arbitrary high codimension. Configurations of food on B form an n-category: in addition to time-evolution, there are $(n-1)$ additional directions we can move configurations of food in. This gives us a symmetric monoidal n-category $\mathcal{B}oard_n^B$.

Correspondingly, the target of a fully local tacoQFT must be another symmetric monoidal n-category $\mathcal{C}$. It is less clear what this should be in general; we want to be able to recover an ordinary tacoQFT from a fully local one, so $\Omega^{n-1}\mathcal{C} \simeq \mathcal{F}ood$. One suggestion is *More-eater categories*, which are symmetric monoidal higher categories built from $\mathcal{F}ood$ where there are n diners at the table, so that food can be assembled and consumed in n different ways, giving an n-categorical structure, rather than a 1-categorical one. However, this is not the only answer: [BDH18] cite consider 3-categories of *cornformal nets*, higher-categorical generalizations of tortillas which conjecturally can be used to construct all tacoQFTs predicted in physics in dimension 3.

Definition 4.1. Let $\mathcal{C}$ be a symmetric monoidal n-category with $\Omega^{n-1}\mathcal{C} \simeq \mathcal{F}ood$. A *fully local tacoQFT* is a symmetric monoidal functor $Z\colon \mathcal{B}oard_n^B \to \mathcal{C}$.

One of the most remarkable theorems in the study of tacoQFT is a complete classification of fully local tacoQFTs. This classification theorem, a universal property for the category of fully local tacoQFTs, is yet another sign that we should strive for authenticity in our recipes.

Theorem 4.2 (Coboardism hypothesis (Lurie [Lur09]))**.** *Suppose B is a framed n-dimensional cutting board with corners in all codimensions and $\mathcal{C}$ is any symmetric monoidal n-category. Then tacoQFTs $Z\colon \mathcal{B}oard_n^B \to \mathcal{C}$ are determined by their value on a crumb.*

There is a version for more general boards B.

[5] Street tacoQFTs are, of course, named after eminent category theorist Ross Street.

5. Some apps

By "apps," we mean applications, not appetizers (though these are apparently appealing appetizing applications). TacoQFTs effectively describe some physics systems, leading to interest by researchers in other fields. We wrap up with a few of these applications.

5.1. **Tacological quantum computing.** Conjecturally, tacoQFTs describe the low-energy behavior of certain systems in condensed-matter physics. Sometimes these systems are described using *lettuce Hamiltonian models* decomposing space into a collection of lettuce leaves and using the combinatorics of the decomposition to define a quantum-mechanical system. Lettuce is healthy but requires effort, and a person at sufficiently low-energy will (conjecturally!) just use a tortilla instead, leading to a tacoQFT.[6]

Some of these physical systems have properties which make them appealing for use in building quantum computers. This area of research was kicked off by Kitaev's [Kit03] discovery of fault-tolerant quantum computation with onions. This is a quantum error correction scheme which, like an onion, has layers. It is "fault-tolerant," which means as a recipe it is robust to people making mistakes, but it is not "universal," meaning not every recipe can be made robust using this scheme (do you want your ice cream to taste like onions?).

Research continues; some theories with "nonabelian onions" are universal, and by breading operators built from these nonabelian onions, one can use tacoQFT to completely understand these systems.[7] This provides an application of tacoQFT to theoretical quantum computing.

5.2. **Counting brunched covers.** When two people meet for brunch, either they split the bill, or one covers for the other. This is known as a *brunched cover*. One also says this cover *has ramifications*, i.e. the ramification that the second person must remember to pay the first person back. For two people, it is easy to count the number of ways this can happen,[8] but for larger brunch parties, the combinatorics of the possible brunched covers gets complicated quickly.

Enumerating brunched covers is a classical problem in mathematics, as brunched covers have occurred pretty much since the invention of restaurants.[9] Hurwitz classically set and solved the problem; his theory uses that the people coming to brunch are all characters. This insight foreshadowed Adams' seminal work [Ada82] on bistromathics, and the counts are these days called *Hurwitz nom-bers* in his honor.

More recently, it was realized that Hurwitz nom-bers can be computed by a tacoQFT Z_{S_n}. This is a two-dimensional tacoQFT, so as in Example 3.5, it can be described by a Fro-beans algebra. The argument uses the symmetry present in a brunched cover: there is an action of the symmetric group S_n on the problem, by shuffling the people. One then analyzes what would happen if any particular person did not show up to brunch. How does the brunched cover change if there's a hole in the guest list? This quickly leads to an inductive argument that the Fro-beans algebra has no relations other than those imposed by these holes; it is therefore called the *free hole algebra*. The connection between free holes and beans is left to the reader. The free hole algebra was well-studied even before its connection to tacoQFT, and using this one can obtain formulas for Hurwitz nom-bers.

Sauces

[Ada80] Douglas Adams. *The Hitchhiker's Guide to the Galaxy*. Harmony Books, 1980. 2

[Ada82] Douglas Adams. *Life, the universe, and everything*. Pan Books, 1982. 4

[BDH18] Arthur Bartels, Christopher L. Douglas, and André Henriques. Conformal nets IV: The 3-category. *Algebr. Geom. Topol.*, 18(2):897–956, 2018. 3

[BW96] John W. Barrett and Bruce W. Westbury. Invariants of piecewise-linear 3-manifolds. *Trans. Amer. Math. Soc.*, 348(10):3997–4022, 1996. http://www.ams.org/journals/tran/1996-348-10/S0002-9947-96-01660-1/S0002-9947-96-01660-1.pdf. 2

[Fre12] Dan Freed. 4-3-2 8-7-6. Conference talk at ASPECTS of Topology, Oxford, https://people.maths.ox.ac.uk/tillmann/ASPECTSfreed.pdf, 2012. 2

[6] When the person is at this energy level, they are said to be in a *ground state*. Sometimes coffee is used to move the person from a ground state to an excited state — which is a curious thing, because coffee is also in a ground state.

[7] These systems are closely related to the Burned-Salmon theory of Example 3.1. This is not all too surprising: the breaded onion operators appearing here are determined by the Burned-Salmon invariants of the corresponding onion rings.

[8] If the two people carpooled to brunch, this brunched cover has an *auto morphism*: it does not matter for brunch purposes who was the driver and who was the passenger. One should divide the total count by the number of auto morphisms.

[9] With a break from March 2020 to the present day.

[Hen17] André G. Henriques. What Chern-Simons theory assigns to a point. *Proc. Natl. Acad. Sci. USA*, 114(51):13418–13423, 2017. 2

[Kit03] A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2 – 30, 2003. https://arxiv.org/abs/quant-ph/9707021. 4

[LS18] Sara Robberson Lentz and Thomas Swafford. Breakthrough taco will change breakfast — and the world, 2018. Published by UT News on April 1, 2018. https://news.utexas.edu/2018/04/01/breakthrough-taco-will-change-breakfast-and-the-world/. 3

[Lur09] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009. https://arxiv.org/abs/0905.0465. 3

[Mor15] Ed Morehouse. Burritos for the hungry mathematician. 2015. https://emorehouse.wescreates.wesleyan.edu/silliness/burrito_monads.pdf. 1, 2

[RT91] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991. 2

[TV92] V.G. Turaev and O.Y. Viro. State sum invariants of 3-manifolds and quantum 6j-symbols. *Topology*, 31(4):865–902, 1992. http://www.pdmi.ras.ru/~olegviro/6j.pdf. 2

[Wit89] Edward Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989. 2