§1 Notations & Definitions

Notation: Let D be a division ring of finite dimension over its center. Note that this makes D a central simple algebra over its center.

Ex: The quaternions over $\mathbb{R}$:

Let $D = \mathbb{H} := \{a + bi + cj + dk \mid a,b,c,d \in \mathbb{R}\}$. The multiplication is:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

We need the following theorems for future classifications:

Wedderburn's little thm: All finite division rings are commutative (& hence fields).

(Frobenius) thm: The only finite dimensional associative division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{H}$, & $\mathbb{C}$.

Def: Assume D has an <u>(anti)-involution</u> $\tau : \begin{cases} \tau(d+d') = \tau(d)+\tau(d') \\ \tau(dd') = \tau(d')\tau(d) \\ \tau(\tau(d)) = d \end{cases}$ $\forall d, d' \in D$

$F' = \{x \in D \mid xy = yx \ \forall y \in D\}$

Def/Notation: Let . Set $F = \{x \in F' \mid \tau(x) = x\}$. F & F' are fields & $F \subseteq F' \subseteq D$.

Def: Let W be an <u>ε-Hermitian space</u>. That is, W is a right D-module equipped with an <u>ε-Hermitian product</u> $\langle \cdot,\cdot \rangle : W \times W \to D$ $\ni$

1) $\langle \cdot,\cdot \rangle$ is sesquilinear w/linearity in the 2^{nd} component.
   i.e. $\langle wd, w'd' \rangle = \tau(d) \langle w, w' \rangle d'$
2) nondegenerate
3) $\langle w', w \rangle = \varepsilon\, \tau(\langle w, w' \rangle)$ (note: the variable swap)

where $\varepsilon \in F'$ & $\varepsilon \tau(\varepsilon) = 1$ (o.w. def'n doesn't make sense).

Def: w & $w' \in W$ are called <u>orthogonal</u> if $\langle w, w' \rangle = 0$

Def: Two ε-Hermition D-spaces are called <u>isometric</u> (resp. <u>similar</u>) if $\exists$ a D-linear bijection from one to the other which preserves the Hermitian product ($\langle Aw, Aw' \rangle = \langle w, w' \rangle$) (resp. up to multiplication by an element of F'). Such a function is called an <u>isometry</u> (resp. <u>similitude</u>)

Def: The set of isometries of $(W, \langle \cdot,\cdot \rangle)$ to itself is a group U, called the <u>unitary group</u> of $(W, \langle \cdot,\cdot \rangle)$.

Remark: 1) A left D-module V is canonically a right D^o-module where $D^o = D$ w/multiplication $d \times d' = d' \cdot d$.
The involution τ permits conversion b/t a right D-module & a left D-module via $d \times v = v\,\tau(d)$ for $v \in V$, $d \in D$.
A sesquilinear form on a left D-module V is linear in the 1^{st} variable & satisfies $\langle dv, d'v' \rangle = d \langle v, v' \rangle \tau(d')$. Inversely, all left D-modules can be converted to right D-modules.

2) $V^* := \mathrm{Hom}(V, D)$ is naturally a left D-module given by $(df)(v) = d \cdot f(v)$.
Similarly V^* has the structure of a right D-module. If D is commutative (aka field), we call V^* the <u>dual of V</u>. The Hermitian product defines a D-iso. b/t W & W^* by $w \to w^*$ where $w^*(v) = \langle w, v \rangle$. This defines an involution: $f \to f^*$ on $A = \mathrm{End}_D W$ where $\langle f(w), w' \rangle = \langle w, f^*(w') \rangle$.
f^* is the <u>adjoint of f</u>. The unitary group is then $U(W) = \{u \in \mathrm{End}_D W \mid uu^* = Id\}$.
It is well understood that the fn. $W \to \mathrm{End}_D W$ induces a bijection b/t

a) Finite dim'l Hermitian spaces, up to similarity
b) Simple central algebras of finite dimension w/involution, up to isomorphism.

§2 Examples of ε-Hermitian Spaces

1) Quadratic Spaces ($D=F$, $\varepsilon=1$). $U(W)$ = orthogonal grp. of $W = O(W)$

2) Symplectic Spaces ($D=F$, $\varepsilon=-1$, char $F \neq 2$). $U(W) = Sp(W)$, symplectic grp of W.

3) Hermitian Spaces ($D=F'$ is a quadratic field extension of F, $\varepsilon=1$)

<u>Fundamental Examples</u>

4) <u>The ε-Hermitian spaces of dimension 1, $D(a)$.</u>
Let $a \in D \ni a = \varepsilon\tau(a)$. Then $D(a)=D$ is a right D-module with ε-Hermitian product $\langle d, d' \rangle = \tau(d)ad'$.

5) <u>The ε-Hermitian hyperbolic plane, H</u>

$H = D \times D$ is a right D-module with ε-Hermitian product
$\langle (d_1, d_2), (d_1', d_2') \rangle = \tau(d_1)d_2' + \varepsilon\tau(d_2)d_1'$.

6) If V is a right D-module, $W = V + V^*$ has an ε-Hermitian product $\langle (v,f), (v',f') \rangle = f'(v) + \varepsilon\tau(f(v'))$. W is an ε-Hermitian space canonically associated to V (generalizing (5))

§3 Involutions

Prop.: An involution τ on D sends the center F' to itself.

Pf: $x,y \in F' \Rightarrow xy=yx \Rightarrow \tau(xy)=\tau(y)\tau(x) = \tau(x)\tau(y)=\tau(yx) \Rightarrow \tau(x),\tau(y) \in F'$. $\boxtimes$
$\tau(yx)=\tau(x)\tau(y)$

(Classification of involutions on simple algebras)

Th: Let τ be an involution on D. Then, there are two cases:

1) $\tau|_{F'} \equiv Id_{F'}$. We say τ is of the 1st Kind (species, type?) (translation unclear). We have either $\varepsilon = 1$ (called Hermitian) or $\varepsilon=-1$ (called anti-Hermitian). Further, $D \cong D^{o}$. Moreover, D admits an involution of the 1st kind iff $D \cong D^{o}$.

2) F' is a separable quadratic field extension of F & $\tau|_{F'}$ is a nontrivial F-automorphism. We say τ is of the 2nd Kind.

Pf of prop. By Hilbert thm 90, if $\varepsilon \in F'$ satisfies $\varepsilon\varepsilon^{\sigma}=1$, then $\exists \mu \in F' \ni \varepsilon = \frac{\mu^{\sigma}}{\mu}$.
So, $\mu\langle w,w'\rangle = \mu\varepsilon\,\tau(\langle w',w\rangle) = \mu^{\tau}\,\tau(\langle w',w\rangle) = \tau(\mu\langle w',w\rangle)$.

Prop: ($\varepsilon=1$) Hence multiplication by μ gives a bijection b/t ε-Hermitian spaces & 1-Hermitian spaces (called Hermitians). Thus, we can limit ourselves to Hermitian spaces when τ is of the 2nd Kind.

Remark: If D^{σ} is the conjugation of D by σ, we have $D \cong (D^{\sigma})^{o}$. Inversely, if $D=(D^{\sigma})^{o}$, then $\exists$ anti-automorphism i on D extending σ. i^2 is then an automorphism. So, $\exists a \in D \ni i^2(d)=ada^{-1}$ $\forall d \in D$. By some thm (citation in book), the element $\alpha := a \cdot i(a) \in F$ does not depend on D & that D admits an involution extending σ iff α is a norm of an element of F'.

- If D is a quaternion space, we have that an involution of the 2nd Kind exists on D iff $D=D^{1} \otimes_F F'$ where D^{1} is a set of quaternions on F.

§4 Involutions on Finite, Local, & Global

1) If F is finite: all finite division algebras are fields. So, either $D=F$ ~~(Hermitian as $\tau|_F = \mathrm{Id}_F$)~~ or $D=F'$ is the unique quadratic extension of F

2) If $F' = \mathbb{C}$, then $D=\mathbb{C}$. The involution is either trivial or the unique, nontrivial automorphism of order 2 on $\mathbb{C}$, complex conjugation

3) If $F' = \mathbb{R}$, then $D = \mathbb{R}$ or $D = \mathbb{H}$.
   a) $D = \mathbb{R}$ does not admit an automorphism of order 2 & so the involution is trivial.
   b) $D = \mathbb{H}$. By the Skolem-Noether thm, the canonical conjugation of $\mathbb{H}$ over $\mathbb{R}$ is multiplication by an inner automorphism. This is the unique involution of the 2^{nd} kind on $\mathbb{H}$. (up to iso.)

4) If F is a local non-Arch. field, we have by similar reasoning:
   a) $D=F$
   b) $D=F'$ is a quadratic separable ext. of F
   c) D is the set of quaternions on F', w/canonical involution being an inner automorphism, & $F'=F$.

There are no others as the condition $D \cong D' \otimes_F F'$ is impossible.