§17 Dual Pairs

(Note: The book uses Z_G to denote the centralizer while I use C_G.)

Def: Let G be a group & $H \leq G$ a subgroup. If the centralizer of the centralizer of H in G is equal to H (i.e. $C_G(C_G(H)) = H$), we call <u>H a Howe subgroup of G</u>.

Def: If H is a Howe subgroup of G & $H' := C_G(H)$, then we say (H, H') is a <u>dual pair</u> in G.

Prop: (H, H') is a dual pair iff $C_G(H) = H'$ & $C_G(H') = H$.

Def: We say that $(Z(G), G)$ is the <u>trivial dual pair</u> of G. Here $Z(G)$ is the center of G.

Prop: 1) $\forall H \leq G$, $C_G(C_G(H))$ is a Howe subgroup of G & $H \leq C_G(C_G(H))$.

2) For any Howe subgroup $K \leq G$, if $H \leq K$, then $C_G(C_G(H)) \leq K$.

Prop: Let G_1, G_2, & G be groups, $H \leq G$, & $H' := C_G(H)$. Suppose that $H, H' \leq G_1 \times G_2 \leq G$. Then, H is a Howe subgroup iff $H = H_1 \times H_2$ where for $i = 1, 2$ H_i is a Howe subgroup of G_i.

Def: We define the <u>product of dual pairs</u> by $(H_1, H_1') \times (H_2, H_2') = (H_1 \times H_2, C_G(H_1 \times H_2))$. The previous proposition shows that this is again a dual pair.

Def: A <u>see saw dual pair</u> is a pair of dual pairs (H, H') & (K, K') such that $H \leq K'$ & $K \leq H'$. We represent these by the diagram:
where the vertical maps are inclusion & the diagonals denote duality.

$$\begin{tikzcd} K' \arrow[d, dash] \arrow[dr, dash] & H' \arrow[d, dash] \arrow[dl, dash] \\ H & K \end{tikzcd}$$

Note: If (H, H') is a dual pair, then (H', H) is also a dual pair. So, for a seesaw dual pair $(H, H'), (K, K')$, we could equivalently define them as $K \leq H$ & $H' \leq K'$ by simply switching H & H'. This viewpoint is taken by some authors (e.g. Kudla).

Dual pair Examples

Ex: For W symplectic

1) $(Sp(W), \{\pm Id\})$ in $Sp(W)$

2) Let e be a standard basis for W & C be the subgroup of isometries which act diagonally with respect to C. Then, (C,C) is a dual pair in $Sp(W)$. (C,C) is irred. iff $\dim_F W = 2$.

Ex: $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL_2$. Then (T,T) is a dual pair in O_2.

See saw Dual pair

1) W symplectic & V orthogonal over F.
Then $(Sp(W), O(V))$ is a dual pair of $Sp(V \otimes W)$.
Suppose $V = V_1 \oplus V_2$ is an orthogonal decomposition. Then,
$O(V_1) \times O(V_2) \subseteq O(V)$ & $(O(V_1)\times O(V_2), Sp(W)\times Sp(W))$ is a dual pair in $Sp(V \otimes W)$.

$$\begin{tikzcd} Sp(W)\times Sp(W) \arrow[d, dash] \arrow[dr, dash] & O(V) \arrow[d, dash] \arrow[dl, dash] \\ Sp(W) & O(V_1)\times O(V_2) \end{tikzcd}$$

is a seesaw dual pair

Howe subrings & subalgebras

Def: Let A be a ring & $B \subseteq A$ a subring. If $C_A(C_A(B)) = B$, then we say B is a Howe subring of A.

Prop: 1) For any subring $B \subseteq A$, $C_A(C_A(B))$ is a Howe subring of A & $B \subseteq C_A(C_A(B))$.

2) For any Howe subring $K \subseteq A$, if $B \subseteq K$, then $C_A(C_A(B)) \subseteq K$.

Cor: For any subring $B \subseteq A$, $C_A(C_A(B)) = \bigcap\limits_{\substack{\text{all Howe} \\ \text{subrings } K \\ \text{with } B \subseteq K \subseteq A}} K$ & $C_A(B)$ is a Howe subring.

These ideas, propositions, & corollaries also make sense for algebras. That is, if we let A & B be algebras now, we obtain the notion of a Howe subalgebra $(C_A(C_A(B)) = B)$ & the above proposition & corollary.

- Our next goal is to classify all Howe subalgebras of simple central algebras. Then, we will classify the Howe-subgroups of the classical groups (unitary groups of type 1 or 2 spaces)

Def: Let W be an ε-Hermitian right D-module of type 1 or 2. A dual pair (H, H') of $U(W)$ is called <u>reductive</u> if

1) H & H' are reductive & (Note: these are equivalent for algebras & probably for groups)
2) W is semisimple as (H,D) & (H',D)-modules

In this case, we call H a <u>reductive Howe subgroup</u> of $U(W)$.

We have similar definitions for reductive dual pairs of $\mathrm{End}_D(W)$.

Def: A dual pair (H, H') of $U(W)$ is called <u>irreducible</u> if there does not exist an orthogonal decomposition of W that is stable under $HH'D$.

We will see later that any reductive dual pair is a product of irreducible reductive dual pairs.

Recall: letting the product on W be null (W is of type 2), we have $U(W) = GL_D(W)$ & orthogonal decompositions are simply direct sums.

Also, W is by definition a right D-module, but we will consider it as a left $\mathrm{End}(W)$-module occasionally. Note that the opposite of D, $D^o \subseteq \mathrm{End}\, W$.

Notation: Given an action of X on a Z-module W (left or right), we denote by $\mathrm{End}_X W$ the set of Z-endomorphisms of W which commute with the action of X.