Global Theta Correspondence (Note: I'm not too familiar with the global case)

F = number field, $\mathbb{A}$ = adeles of F, $\psi: \mathbb{A}_F \to \mathbb{C}^\times$ nontrivial character.
$W = X \oplus Y$ symplectic v.s. over F w/ X, Y maximal isotropic subspaces.
Schwartz space $\rightarrow$ $S(X(\mathbb{A})) = \bigotimes_v S(X(F_v)) \Rightarrow \exists$ projective rep. of $Sp(W)(\mathbb{A})$ on $S(X(\mathbb{A}))$
by taking tensor products of local rep's ~~Weil rep~~.
This projective rep. becomes an ordinary rep. of $\widetilde{Sp}(W)(\mathbb{A})$.

(Weil) th: The covering $\widetilde{Sp}(W)(\mathbb{A}) \to Sp(W)(\mathbb{A})$ splits over $Sp(W)(F)$.

Thus $Sp(W)(F)$ acts on $S(X(\mathbb{A}))$. Define a distribution Θ on $S(X(\mathbb{A}))$ by $\Theta(\phi) = \sum_{x \in X(F)} \phi(x)$. This distribution is $Sp(W)(F)$ invariant.
So, the function $g \mapsto \Theta(g \cdot \phi) := \Theta_\phi(g)$ defines a function

$$\Theta_\phi : Sp(W)(F) \backslash \widetilde{Sp}(W)(\mathbb{A}) \to \mathbb{C}.$$ These are called <u>theta functions</u>.
They are slowly increasing & hence automorphic.

Let (G_1, G_2) be a dual reductive pair in $Sp(W)$ which is defined over F.
Let π be a cuspidal rep of G_1, realized on a space of cusp form on $G_1(F)\backslash G_1(\mathbb{A})$.
For $\phi \in S(X(\mathbb{A}))$ & $f \in \pi_1$, define

$$\Theta_\phi(f)(g_2) := \int_{G_1(F)\backslash G_1(\mathbb{A})} \Theta_\phi(g_1, g_2) f(g_1)\, dg_1 .$$

$\Theta_\phi(f)$ is an automorphic form on G_2 & is called the <u>theta lift</u> of f.

Questions. (1) When is the space generated by $\Theta_\phi(f)$ ~~not identi~~ nonzero?
(2) What is the relation of the span of $\Theta_\phi(f)$ to the local theta correspondence?
(3) What about towers of lifts?

towards (2)

(Rallis, 1984) th: If $\Theta_\phi(f)$ consists of cusp forms, then they generate an irreducible space that is isomorphic to $\bigotimes_v \Theta_v(\tilde{\pi}_v)$ where $\tilde{\pi}_v$ is the contragradient of π_v.

towards (3) &(1)

th: Consider Θ lifts from $Sp(W)$ to various $O(V + n\mathbb{H})$ where $\mathbb{H}$ is a ~~hyperplane~~ hyperbolic plane.
1) Let π be a cuspidal automorphic rep. of $Sp(W)$ & $\Theta_n(\pi)$ be its lift to $O(V+n\mathbb{H})$.
Let n_0 be the smallest integer ≥ 0 $\ni$ $\Theta_{n_0}(\pi) \neq 0$. Then $\Theta_{n_0}(\pi)$ is cuspidal & $\Theta_n(\pi) \neq 0$ & are noncuspidal for $n > n_0$

(th of Rallis also; same setup as before)

2) $\Theta_n(\pi) \neq 0$ for $n \geq \dim W$

Applications

Global theta correspondence b/t $PGL(2)$ & $\overline{SL}(2)$

(Waldspurger) th: 1) For a cuspidal automorphic rep. π of $PGL(2)$, $\Theta_\pi \neq 0$ on $\overline{SL}(2)$ iff $L(\pi, \frac{1}{2}) \neq 0$

2) For an automorphic rep. π of $\overline{SL}(2)$, $\Theta_\pi \neq 0$ iff π has a Ψ-Whitaker model

3) $\forall$ automorphic rep. π of $PGL(2)$, $\otimes \Theta_v(\pi_v)$ is automorphic iff the sign in the functional equation for the L-function of π, $\varepsilon(\pi, \frac{1}{2}) = 1$.

Other applications include functoriality, Siegel-Weil formulas, cohomology of Shimura varieties