Classification of Howe sub-algebras of Simple Central Algebras

th: 1) Any reductive Howe subalgebra B of a simple central algebra is a product of irreducible Howe subalgebras B_i of central simple algebras.
2) For any tensor product decomposition $W = W_1 \otimes_{D_1} W_2$, the pair $(\mathrm{End}_{D_1} W_1, \mathrm{End}_{(D_1,D)} W)$ is an irreducible dual pair of $\mathrm{End}_D W$
3) Any irreducible dual pair is of the form (2).

Remarks: 1) This generalizes a theorem of H. Weyl: Any simple subalgebra B of a central simple algebra A has $B = C_A(C_A(B))$.
2) A reductive dual pair of $\mathrm{End}_D(W)$ is also a dual pair in $\mathrm{End}_F(W)$.

Pf: Let $A = \mathrm{End}_D(W)$ (A has this form by remark (2) in §1). Let $B \subseteq A$ be subalgebra which operates semi-simply on W. Then $W = \oplus m_i V_i$ where each V_i is a simple (B,D)-submodule of W & are inequivalent by action of B. Schur's lemma states that any B-homomorphism between simple modules is either 0 or invertible. $V_i \not\cong V_j$ $\forall i \neq j$ & so $\mathrm{Hom}(V_i, V_j) = 0$. Also, any nonzero B-homomorphism $V_i \to V_i$ is invertible. So, $\mathrm{End} V_i$ is a division ring. Consequently, $C_A(B) \cong \oplus M(m_i, D_i)$ where D_i is a division ring (B-hom's are those which commute with the action of B & hence are $C_A(B)$). Thus, $C_A(B)$ is reductive & operates semi-simply on W. In particular, we see that the statements (i) + (ii) in our definition of reductive dual pairs are equivalent.

1) Let (B, B') be a dual reductive pair of A. Then B & B' operate semi-simply on W. As above, decompose $W \cong \oplus m_i V_i$. Let $A_i := \mathrm{End}_D m_i V_i$. Then, B & B' identify canonically with their images B_i, B_i' in A_i. As the V_i's are simple, (B_i, B_i') form irreducible dual pairs whose product is (B, B').

2) Let $B = \mathrm{End}_{D_1} W_1$ & $B' = \mathrm{End}_{(D_1,D)} W_2$. It is simple to check B & B' commute. Let $A = \mathrm{End}_D W$. Let Y be a basis of B on D_1 & Y' be a basis of B' on $D_1^\circ \otimes_F D$ $\ni$ $\{y \otimes_{D_1} y'\}_{\substack{y \in Y \\ y' \in Y'}}$ forms a basis for A.
↑ opposite of D_1

Now $u \in A$ commutes with B iff $u = \sum f(y') \otimes_{D_1} y'$ where f, is a function from Y' to the center of B. This center is contained in $D_1^\circ \otimes_F D$ & hence $C_A(B) \subseteq B'$. Thus, $C_A(B) = B'$. Swapping B & B' shows $C_A(B') = B$.
↑ $B' \subseteq C_A(B)$ trivially

Hence (B, B') is indeed a dual pair. ~~The result of (3) will give that~~ Furthermore, (B, B') is irreducible.

3) Suppose that W is (BB', D)-irreducible. Then, W is (B', D)-isotypic. That is, $W = mW'$ where W' is (B', D)-irreducible. So, $\mathrm{End}_{(B',D)} W'$ is a division ring D_1 (by Schur's lemma) whose center contains the center of D. Now, $B' = \mathrm{End}_{(D_1, D)} W'$. We can write $W = W_1 \otimes_{D_1} W_2$ where W_1 is a right D_1-module of dimension m & $W_2 = W'$. Then $C_A(B')$ for $A = \mathrm{End}_D W$ is $B = \mathrm{End}_{D_1} W_1$ & $C_A(B) = B' = \mathrm{End}_{(D_1, D)}(W_2)$ ($= \mathrm{End}_{D_2} W_2$ where D_2 is defined as in §16). $\square$

Connections b/t Irred. dual pairs of Groups & Algebras

Lemma: Set $G := U(W)$ where W is of type 1 or 2. If (H, H') is an irred. dual pair of G, then the algebras $(B = \mathrm{End}_{DH} W, B' = \mathrm{End}_{DH'} W)$ form an irred. dual pair of $A = \mathrm{End}_D W$ & $B \cap G = H'$, $B' \cap G = H$. ← (book is wrong here)

(since $B' = \mathrm{End}_{DH'}(W)$)

Pf: Note $B' \subseteq C_A(C_A(B'))$ in general. Also, $\forall h' \in H'$ & $b' \in B'$, we have $b'(h'(w)) = h'(b'(w))$. So, $H' \subseteq C_A(B')$. Let $f \in C_A(C_A(B'))$. Then $fg = gf$ $\forall g \in C_A(B')$. Hence $f(\partial h' w) \underset{(f \in \mathrm{End}_D W = A)}{=} \partial f(h' w) \underset{\left(\substack{h' \in H' \subseteq C_A(B') \\ + f \in C_A(C_A(B'))}\right)}{=} \partial h' f(w)$ $\forall \partial \in D, h' \in H', w \in W$. Hence $f \in \mathrm{End}_{DH'} W = B'$. So, $B' = C_A(C_A(B'))$ & thus B' is a Howe subalgebra of $\mathrm{End}_D W = A$.

(I should be doing ∂ on the right)

If $f \in B' \cap G$, then $f(h'(w \cdot \partial)) = h'(f(w)) \cdot \partial$ $\forall h' \in H', \partial \in D$. So, f commutes with H' & hence (as $f \in G = U(W)$) $f \in C_G(H') = H$. Any $h \in H$ also satisfies $h(h'(w \cdot \partial)) = h'(h(w)) \cdot \partial$ & hence $h \in B' \cap G$. Thus $B' \cap G = H$. It follows similarly that (B, B') forms a dual pair & $B \cap G = H'$.

(Fixed ∂ here)

Now, suppose (B, B') is reducible. Then, $\exists$ an orthog. decomp. of W that is stable under (B, B'). As $H \subseteq B'$ & $H' \subseteq B$, this orthog. decomp. is stable under (H, H'). But, (H, H') is irred $\lightning$ So, (B, B') is irred. $\boxtimes$

Note: The converse of this lemma is not true. For example, let K/F be a separable finite extension of F & G be the orthogonal group associated to the quadratic form $tr_{K/F}(x^2)$. Then $B = B' = K$ is a dual pair in $\mathrm{End}_F K$. But $K \cap G = \{\pm I \partial\}$ is not its own centralizer in G:

$C_G(\{\pm I \partial\}) = G \neq \{\pm I \partial\}$.

Classification of Howe Reductive Subgroups for Classical Groups

th: 1) Any reductive dual pair of $U(W)$ is a product of irred. reductive dual pairs.
2) Any reductive irred. nontrivial dual pair of $U(W)$ is isomorphic to one of the following:

a) $(U(W_1), U(W_2))$ for any decomposition $W = W_1 \otimes_{D'} W_2$ where each factor is not of the following types:
- orthogonal hyperbolic of dim. 2 on $D' = \mathbb{F}_3$
- Anti-Hermitian of dim. 1 on quaternions D' with $D = F$.

b) $(GL_{D_1}(X_1), GL_{D_2}(X_2))$ if W is totally isotropic & non-degenerate (type 1) where X is a Lagrangian subspace of W with decomposition $X = X_1 \otimes_{D'} X_2$.

Remark: The algebra generated by $U(W)$ in $A = End_D W$ is A except in the excluded cases of (a).

Pf: (1) follows simply by definition of products of pairs.
(2) If W is of type 2, then the claim follows from the classification for simple central algebras. Thus, suppose W is of type 1 (non-degenerate ε-Hermitian space). Assume (H,H') is an irred. reductive dual pair of W. Furthermore, assume that
(b)(direct) W has no non-degenerate subspace fixed by H or H' (so not of case (a)). Then, it is possible that there is a right D-submodule $X \subseteq W \ni X$ is fixed & $X^\circ := X \cap X^\perp \neq \{0\}$. (aka X doesn't give an orthog. decomp)
Then, X° is totally isotropic & X° is fixed by HH'. Let $P(X^\circ)$ (parabolic) be the stabilizer of X° in $U(W)$. Since (H,H') is reductive, HH' is reductive & so $HH' \cap (\text{unipotent radical of } P(X^\circ)) = \{Id\}$. Thus, we identify (H,H') with a reductive dual pair (K,K') of a Levi subgroup M of $P(X^\circ)$.
Now, $M \cong GL_D(X^\circ) \times U(W')$ where W' is non-degenerate or 0.
If $W' \neq 0$, then (K,K') is not irred. in $U(W)$ & hence neither was (H,H').
If $W' = 0$, then W is hyperbolic, $X = X^\circ$ is a Lagrangian, & (K,K') is an irred. dual pair of $GL_D(X)$. ~~So, any irred. dual pair is of the form (b) in our case.~~
~~Conversely, let (H,H') be an irred. dual pair of $GL_D(X)$.~~

(Pf continues next page)

(note: X Lagrangian of $W \Rightarrow W \cong X \oplus X^*$)

b) converse: Now, suppose (H,H') is an irred. reductive dual pair of $GL_D(X)$. Then, (H,H') is a dual pair in $U(W)$ if X is a Lagrangian of W. Also, $GL_D(X)$ includes naturally into $U(W)$:

Let $X = X_1 \otimes_D X_2 \ni (H,H') = (GL_{D_1}(X_1), GL_{D_2}(X_2))$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(W)$ where $a \in GL_D(X)$, $d \in GL_D(X^*)$, $b \in Hom_D(X^*, X)$, & $c \in Hom_D(X, X^*)$ commutes with $H \Rightarrow b(h^*)^{-1} = hb$, then $\ker b$ & $\operatorname{Im} b$ are H-invariant. The canonical decomposition of h gives a bijection $\eta: X_1 \otimes_{D_1} Y_2 \cong (X_1 \otimes_{D_1} Y_2)^*$ where $Y_2 \subseteq X_2$ & $b(h^*)^{-1} = hb$. $\exists b$ of reduced norm on $F \ni \det_F b \neq \pm 1$. Since the reduced norm is multiplicative, we have $Y_2 = \{0\}$ & hence $b = 0$. By similar reasoning $c = 0$. So, $C_{U(W)}(H) = H'$ is an irred. reductive dual pair of $U(W)$.

(a) Suppose W has no (non-degenerate or o.w) stable subspace under (H,H'). By the previous lemma, $\exists$ a decomp. $W = W_1 \otimes_{D_1} W_2 \ni H = U(W) \cap B$ + $H' = U(W) \cap B'$ where $B = End_{D_1} W_1$ & $B' = End_{(D_1, D)} W$. Now, $U(W)$ is stable under the adjoint involution & hence so are B & B'. By the bijection between ε-Hermitian spaces & algebras with involutions, we have that for $i = 1, 2$, W_i is a right D_i-module ε_i-Hermitian with product $\langle \cdot, \cdot \rangle_i$ defined up to similitude & that $H = U(W_1)$ & $H' = U(W_2)$.

(converse): Conversely, any decomposition of W as an ε-Hermitian tensor product results in dual pairs except in the excluded cases. $\square$

As irred. (H,H') either fixes a degenerate subspace or none at all, this completes the classification.