II Lagrangians (char $\neq 2$)

Let W be a type 1 or type 2 space (nondegenerate ε-Hermitian or 0 product ~~degenerate ε-Her~~)

Def: We say $X \subseteq W$ is a <u>Lagrangian</u> of W if

(type 1) X is a maximal totally isotropic submodule of W & W is hyperbolic

(type 2) X is a nonzero submodule & W is of type 2.

Notation: $\Omega = \Omega(W)$ is the set of Lagrangians on W.
$\Omega(r)$ is the set of Lagrangians of dimension r.

Prop:
1) $\Omega = \emptyset$ if W is of type 1 & non-hyperbolic
2) If W is hyperbolic of Witt index m, then $\Omega = \Omega(m)$.
3) $\Omega(r)$ is the set of Grassmanian subspaces of W if W is of type 2
4) $U(W)$ acts on Ω. The orbits of this action are $\Omega(r)$. It is transitive if W is of type 1.

Def: If $W \cong mH$ we say $X+Y$ is a <u>complete polarization of W</u> if X & Y are Lagrangians & $W = X+Y$.

Notation: Let V be a right D-module. Set $S^2(V,\varepsilon)^*$ to be the set of sesquilinear forms on V that satisfy the symmetry property in our definition of ε-Hermitian product.
Note: these forms are allowed to be degenerate.

(Parameterization of Ω by polarizations)

Lemma: Let $W = mH$ & $X+Y = W$ be a complete polarization. Then, $X+Y$ induces a natural parameterization on Ω. Specifically, there is a canonical bijection
$$\Omega \leftrightarrow \bigcup_{V \in \Omega(X)} S^2(V,-\varepsilon)^*.$$

Pf: Let $Z \in \Omega$. Let π be the projection onto X. $\forall z,z' \in Z$, set $B(z,z') := \langle \pi(z), z' \rangle$.
Write $z = x+y$, $z' = x'+y'$ (by our complete polarization). Since Z is Lagrangian,
$$0 = \langle z,z' \rangle = \langle x,x' \rangle + \langle x,y' \rangle + \langle y,x' \rangle + \langle y,y' \rangle \underset{(X,Y \text{ Lagrangian})}{=} \langle x,y' \rangle + \langle y,x' \rangle = \langle x,y' \rangle + \underbrace{\varepsilon\tau(\langle x',y \rangle)}_{*}$$
Hence $\langle x,y' \rangle = -\varepsilon\tau(\langle x',y \rangle)$.

~~Also, $0 = \langle z',z \rangle = \langle x',x \rangle + \langle x',y \rangle + \langle y',x \rangle + \langle y',y \rangle = \langle x',y \rangle + \langle y',x \rangle$ so $\langle x',y \rangle = -\langle y',x \rangle$. Hence $\langle x,y' \rangle = -\varepsilon\tau(\langle x',y \rangle) = -\varepsilon\tau(-\langle y',x \rangle) = \varepsilon\tau(\langle y',x \rangle)$~~

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(PF of lemma cont...)

$$\text{So, } B(z,z') = \langle x, x'+y' \rangle \overset{x \text{ Lagrangian}}{=} \langle x, y' \rangle = -\varepsilon \tau(\langle x', y \rangle) = -\varepsilon \tau(\langle x', x \rangle + \langle x', y \rangle)$$
$$= -\varepsilon \tau(\langle x', x+y \rangle) = -\varepsilon \tau(\langle \pi(z'), z \rangle) = -\varepsilon \tau(B(z', z)).$$

So, B induces a $-\varepsilon$-Hermitian form on $V = \pi(Z)$. Hence $B \in \bigcup_{V \in \Omega(X)} S^2(V, -\varepsilon)$.

Conversely, if $B \in \bigcup_{V \in \Omega(X)} S^2(V, -\varepsilon)$ set $Z := \{x+y \mid \forall x' \in V,\ \langle x', y \rangle = B(x', x)\}$.

It is straightforward to check that Z is Lagrangian. $\square$

Now, let $W = mH$ & suppose $W = W_1 \oplus (-W_2)$ is an orthogonal decomposition of W in ε-Hermitian spaces.

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Lemma: Assume the above ($W = mH = W_1 \oplus (-W_2)$). Then there is a bijection
$\Omega \leftrightarrow \{(Z_1, Z_2, \Phi) \mid Z_i$ is an isotropic subspace of W_i & Φ is an isometry $Z_1^\perp / Z_1 \cong Z_2^\perp / Z_2\}$.

Let Z be a Lagrangian of W ($Z \in \Omega$), $\pi_1(Z)$ its projection onto W_1 paralleling W_2, & $Z_1 = Z \cap W_1$

$Z_1^\perp = \pi_1(Z)$. $\quad$ ($Z_1^\perp = \{u \in W_1 \mid \langle u, z_1 \rangle = 0 \ \forall z_1 \in Z_1\}$)

Let r_i & n_i be the Witt index & dimension of W_i, respectively. Then, $W_i \cong r_i H \oplus W_i^0$
$n_i = 2r_i + n_i^0$ where n_i^0 is the dimension of W_i^0, the anisotropic part of W_i. We have $n_1^0 = n_2^0$ only if $W_1^0 \cong W_2^0$ (the Witt classes of W_i are the same). Set $d = \dim Z$, $d_i = \dim Z_i$, & $\lambda_i := \dim Z_i^\perp - \dim \pi_i(Z)$. Certainly, $\pi_i(Z) \subseteq Z_i^\perp$ as Z is a Lagrangian. So, $\lambda_i \geq 0$.

Now,
$$d = r_1 + r_2 + \overbrace{n_1^0}^{\text{I don't think I need this term}} = \dim(\pi_1(Z) + Z_2) = d_1 + 2(r_1 - d_1) + n_1^0 - \lambda_1 + d_2.$$

Hence $(r_1 - d_1) - (r_2 - d_2) = -\lambda_1$. By symetry, $(r_1 - d_1) - (r_2 - d_2) = \lambda_2$. So, $\lambda_2 = -\lambda_1$. But $\lambda_i \geq 0$.
Hence $\lambda_1 = 0 = \lambda_2$. This proves $\boxed{\text{I}}$

Now, for $z = z_1 + z_2,\ z' = z_1' + z_2' \in Z$, we have $\langle z, z' \rangle \overset{Z \text{ Lagrangian}}{=} 0$. So, $\langle z_1, z_1' \rangle - \langle z_2, z_2' \rangle = 0$.
This gives a correspondence between $Z_1^\perp$ & $Z_2^\perp$ from Z (using $\boxed{\text{I}}$). This correspondence induces the isomorphism Φ

Inversely, let (Z_1, Z_2, Φ) be such a triplet. Set $Z = \{z_1 + z_2 \in Z_1^\perp + Z_2^\perp \mid \Phi(z_1 + Z_1) = z_2 + Z_2\}$.
Then, Z is a Lagrangian of W & this construction is the inverse of the previous. $\square$

Assume the same setup as previous lemma.

If $r_1 \le r_2$, then $d_i = \dim Z_i$ satisfy $0 \le d_1 \le r_2$, $d_2 = d_1 + (r_2 - r_1)$. Let $U = U(W)$, $U_i = U(W_i)$. Then $U(W_i) \hookrightarrow U(W)$ by extending $U(W_i)$ trivially across W_j ($j \ne i$, trivial means $f(w_j) = w_j$). Then, $u_1 u_2 = u_2 u_1$ $\forall u_i \in U_i$. The action of $U_1 U_2$ on Ω is given by (using previous lemma)

$$(u_1, u_2) \cdot (Z_1, Z_2, \Phi) = (u_1 Z_1, u_2 Z_2, u_2 \Phi u_1^{-1}).$$ This gives the following:

Lemma: If $r_1 \le r_2$, two Lagrangians Z & Z' are in the same orbit under $U_1 U_2$ iff $\dim Z_1 = \dim Z_1'$.

Also, there are $r_1 + 1$ orbits. The stabilizer of (Z_1, Z_2, Φ) in $U_1 U_2$ is $\{u_1 u_2 \in P_1(Z_1) P_2(Z_2) \mid u_2 = \Phi u_1 \Phi^{-1} \text{ on } Z_2^{\circ}/Z_2\}$.

Remark: If W is of type 2, then $W = W_1 \oplus W_2$ is a direct sum induced by a parameterization of Grassmanians $\Omega(r)$. The previous lemmas are the ϵ-Hermitian analogs of this.

Lemma: There is a bijection $\Omega(r) \leftrightarrow \{(Z_1, T_1, Z_2, T_2, \Psi) \mid Z_i \subseteq T_i \subseteq W_i,\ \Psi \text{ is an iso. } T_1/Z_1 \cong T_2/Z_2\}$.

If W is type 2.

The dimensions d_i, e_i of Z_i, T_i satisfy $r = e_2 + d_1 = e_1 + d_2$, $e_i, d_i \le r_i$ ($r_i = \dim W_i$).

They are invariants of the orbits of $\Omega(r)$ under action $U_1 U_2$.

The stabilizer of $(Z_1, T_1, Z_2, T_2, \Psi)$ in $U_1 U_2$ is $\{g_1 g_2 \in P_1(Z_1 \subseteq T_1) P_2(Z_2 \subseteq T_2) \mid g_2 = \Psi g_1 \Psi^{-1}\}$.

Idea of Pf: If $Z \in \Omega(r)$ with $r \le n_1 + n_2$, set $Z_i = W_i \cap Z$ & $T_i = \pi_i(Z)$.

Some Geometric Lemmas

(type 1) Let W be of type 1. Two elements $(w_i), (v_i) \in mW$ $(m \geq 1)$ are in the same orbit under the action of $U(W)$ iff their coordinates have the same Gram matrix (or moment matrix), meaning $(\langle w_i, w_j \rangle)_{ij} = (\langle v_i, v_j \rangle)_{ij}$ & generate subspaces of the same dimension. (consequence of Witt's thm).

Equivalently, let V be a right D-module of dimension m. ~~$Hom_D(V,W)$~~ $U(W)$ acts on $Hom_D(V,W)$ via $(f,u) \mapsto u \circ f$ where $u \in U(W)$ & $f \in Hom_D(V,W)$. Let $g \in Hom_D(V,W)$. Then $g = uf$ for some $f \in Hom_D(V,W)$, $u \in U(W)$ iff $Z = \ker g = \ker f$ & $V/_{\ker f} \cong V/_{\ker g}$ are isometric for the forms induced by $\langle \cdot, \cdot \rangle_W$.

Note: these induced forms may be degenerate. Hence we have the following lemma:

Lemma: There is a bijection b/t the orbits of $Hom(V,W)$ under the action of $U(W)$ & the set of couples $\left\{ (Z,B) \mid Z \subseteq V \text{ submodule}, B \text{ is a } \cancel{\text{non}}\text{degenerate } \varepsilon\text{-Hermitian form or non-degen. on } Z \ \& \ (V/Z, B) \text{ is isometric to a subspace of } W \right\}$

Note: If "$\overset{\text{Witt index of } W}{r} \geq \overset{\dim V}{m}$", then $(V/Z, B)$ is isometric to a subspace of W automatically.

The $U(W)$-orbits of $Hom_D(V,W) \times Hom_D(W,V')$ where V & V' are right D-modules of finite dimensions m & m', respectively are determined by the previous lemma ~~given by~~ due to the isomorphism $W \cong W^*$ given by the product.

Note: The action of $U(W)$ on $Hom_D(V,W) \times Hom_D(W,V')$ is $u(f,g) = (uf, gu^{-1})$.

ype 2) Suppose now that W is of type 2 & V, V' are as above. The invariants of a $U(W)$ orbit of $Hom_D(V,W) \times Hom_D(W,V')$ are $Z = \ker f$, $Z' = \operatorname{Im} g$, $\varphi = gf$.

mma: The $U(W)$ orbits of $Hom_D(V,W) \times Hom_D(W,V')$ are in bijection with the set of triplets $\left\{ (Z, Z', \varphi) \mid Z, Z' \text{ are submodules of } V, V' \ \& \ \varphi \in Hom(V/Z, Z') \ni \cancel{\dim V/Z, \dim Z', \dim(\ker\varphi) + \dim Z' \leq} \right.$
$\left. \dim V/Z, \dim Z', \dim \ker\varphi + \dim Z' \leq \dim W \right\}$.

Note: The dimensions condition is immediate if $m + m' \leq n$.

Fixed Lagrangians of a reductive Howe subgroup

Suppose (U_1, U_2) is an irreducible reductive dual pair of $U(W)$. Let $\Omega^1 \subseteq \Omega$ be the subset of Lagrangians of W which are fixed by U_1. Suppose that $(U_1, U_2) = (U(W_1), U(W_2))$ where $W = W_1 \otimes_D W_2$. Let Ω_2 be the Lagrangians of W_2.

Note: Ω^1 & Ω_2 may be empty.

Lemma: We have a bijection between Ω^1 & Ω_2 that is compatible with the action of U_2 if W_1 is not an orthogonal hyperbolic plane on $\mathbb{F}_3$.

Pf: Suppose W is type 1 & W_1 is not an orthogonal hyperbolic plane on $\mathbb{F}_3$. Any invariant ~~subspace~~ submodule of W under U_1 is of the form $W_1 \otimes Z'$ where Z' is a submodule of W_2 ~~(this fact follows from the next lemma)~~. Now, $W_1 \otimes Z'$ is isotropic iff Z' is isotropic (as W_1 is non-degenerate) $W_1 \otimes Z'$ is a Lagrangian iff $\dim Z' = \frac{\dim W_2}{2}$ (= Witt index of W_2) & W_2 is hyperbolic.

Thus, $W_1 \otimes Z' \in \Omega^1$ iff $Z' \in \Omega_2$.

Now, suppose W is of type 2. Any invariant ~~subspace~~ submodule of W under U_2 is of the form $(W_1 \otimes Z') + (W_1^* \otimes Z'')$ where Z' & Z'' are submodules of W_2. $(W_1 \otimes Z') + (W_1^* \otimes Z'')$ is Lagrangian iff Z'' is orthogonal to Z' in W_2^* (o.w. we don't obtain a submodule of W). $\square$

Lemma: The commutant of $U(W)$ in $\operatorname{End} W$ is

a) isomorphic to $\mathbb{F}_3 \times \mathbb{F}_3$ if W is the orthogonal hyperbolic plane on $\mathbb{F}_3$

b) equal to $\operatorname{End} W$ if W is orthogonal of dimension 1

c) isomorphic to D otherwise.

I don't know how to translate this. However, k is the division subring generated by $1 \in D$. Specifically, $k \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } \operatorname{char} D = p \\ \mathbb{Q} & \text{o.w.} \end{cases}$

(Following Dieudonné)

Pf: (c) Let $A = \operatorname{End} W$ be the set of endomorphisms of the $\mathbb{Z}$-module W. If k is the 1st subring of D, then $\operatorname{End} W = \operatorname{End}_k W$. If $z \in A$ commutes with $h \in A$, then z stabilizes the submodule of h-fixed points. If $z \in A$ commutes with $U(W)$, then in particular, it commutes with the symmetries & transvections (skew maps) of $U(W)$ & leaves stable the nonisotropic hyperplanes of W (i.e. the restriction of the product on W remains non-degenerate), & if W is not orthogonal, then z stabilizes the isotropic submodules also.

Hence if W is anisotropic or not orthogonal, z stablizes all submodules of W. If W is orthogonal of dimension ≥ 3, we get that all isotropic are the intersection of 2 nonisotropic planes & hence z stabilizes all submodules of W again. (Pf continues next page)

(PF cont...)

If $\dim_D W > 1$ & z stabilizes the right D-submodules of W, then $\exists\, d \in D \ni z(w) = w \cdot d\ \forall w \in W$. Conversely, any z of this form commutes with $U(W)$. Therefore $\operatorname{End} W \cong D$ which proves some of (c)

(a) Suppose now that W is an orthogonal hyperbolic plane. Let $\{e, f\}$ be a hyperbolic basis, On this basis $U(W)$ is represented by diagonal & antidiagonal matrices $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \begin{pmatrix} 0 & a \\ 1/a & 0 \end{pmatrix} \ni a \in F^{\times}$. Then $z(xe + yf) = A(x,y)e + B(x,y)f$ where $x, y \in F$ &
$\forall A, B : F \times F \to F \ni A(x,y) = B(y,x)$, $A(xa, y/a) = a A(x,y)$, & $A(x,y) = A(x,0) + B(0,y)$ where $a \in F$.
Set $\alpha = A(1,0)$ & $\beta = A(0,1)$. Then $z(xe + yf) = \alpha(xe + yf) + \beta(e/y + f/x)$.
If $\exists\, a \in K \ni a^2 \neq 1$, then z is a K-endomorphism of W iff $\beta = 0$ & hence the centralizer of $U(W)$ in $\operatorname{End} W$ is F. Otherwise ($\nexists\, a \in K \ni a^2 \neq 1$), $K = \mathbb{F}_3$ & the centralizer of $U(W)$ in $\operatorname{End} W$ is $\mathbb{F}_3 \times \mathbb{F}_3$ by direct verification. This proves (a).

(b) Suppose $\dim_D W = 1$. Let $W = D(a)$ where $\tau(a) = \varepsilon a$. Then $U(W) \cong D^a = \{d \in D \mid d a \tau(d) = a\}$.
Let $E(a)$ be the subring of D generated by D^a & the 1^{st} sub of D. The commutant of $U(W)$ in $\operatorname{End} W$ is $\operatorname{End}_{E(a)} W$. $\forall K \subseteq D$, D is the commutant of $U(W)$ in $\operatorname{End}_K W$ iff $K \cdot E(a) = D$.

1) If τ is trivial on D, then W is orthogonal, $U(W) = \{\pm \mathrm{Id}\}$ & the commutant of $U(W)$ in $\operatorname{End} W$ is $\operatorname{End} W$. This proves (b).

2) If τ is nontrivial & a is in the center of D, then $U(W) \cong \{d \in D \mid d\tau(d) = 1\}$ does not depend on a & $E = E(a)$ does not depend on a also. ~~It follows that End W is again the commutant of U(W) in End W. This proves (b).~~

We now examine some more cases

- If D is finite, then $D = F'$ & of degree 2 over F. By the next lemma, $E = F' = D$. Here the commutant of $U(W)$ in $\operatorname{End} W$ is $F' = D$.
- If D is local nonArchimedean, then $D = F'$ or D is a quaternionic ring & W is Hermitian. By the next lemma, $D = E$ if $D = F'$. If D is quaternionic, then E contains every maximal subfield of D & hence $E = D$. So, again, the commutant of $U(W)$ in $\operatorname{End} W$ is D.
- If D is quaternionic & has a pure quaternion (i.e., Real part $= 0$), then $F(a) \subseteq E(a)$. One can check that $E(a)$ is not commutative & hence $E(a) = D$. So, again, the commutant of $U(W)$ in $\operatorname{End} W$ is D. This proves (c) in full & the lemma. $\square$

Lemma: Let $F \subseteq F'$ be a quadratic field extension where $\operatorname{char} F \neq 2$ & F is either finite or local non Archimedean. Let $\bar{F}$ be an algebraic closure of F. Then there does <u>not</u> exist a nontrivial homomorphism $F' \to \bar{F}$ that is trivial on the units of F' of norm 1 in F.

(Won't prove)

★ **Cor:** If W is not the orthogonal hyperbolic plane on $\mathbb{F}_3$, then there does <u>not</u> exist a right D-submodule $V \subseteq W$ which is stable under $U(W)$.

Pf: Suppose that V is stable. Then V is not totally isotropic (unless W is degenerate in which case the cor. is simple)

Indeed if V is totally isotropic, then we can obtain a hyperbolic basis $\{e_i, f_j\}_{i,j=1}^{\dim V}$ for $\underbrace{V \oplus V^*}_{\cong \dim V \cdot H} \subseteq W$ (Witt decomp.) satisfying $\langle e_i, f_j \rangle = \delta_{ij}$. The map sending $e_i \mapsto f_i$ & $f_i \mapsto e_i$, $w \mapsto w$ $\forall w \in W \setminus V$ is an isometry of W that does <u>not</u> fix V (it maps V to V^*).

Now, V is stable & not totally isotropic. It is simple to check that $V^\perp$ is also stable. Hence $V^0 = V \cap V^\perp$ is stable. But V^0 is totally isotropic & we've seen that stable totally isotropic subspaces are 0. Therefore, V is not isotropic & $W = V \oplus V^\perp$ & hence $U(W) = U(V) \times U(V^\perp)$. By the previous lemma,

$Id_{\dim V}$ eally

the commutant of $U(V)$ in $\operatorname{End} V$ is $\mathbb{F}_3 \times \mathbb{F}_3$, $\operatorname{End} V$, or D & similarly for $U(V^\perp)$ in $\operatorname{End} V^\perp$. Also, the commutant of $U(W)$ is D (if $\dim W = 1$ its trivial). Thus, we have (extending $\operatorname{End} V$ or $\operatorname{End} V^\perp$ identically across $V^\perp$ or V respectively)),
$(\mathbb{F}_3 \times \mathbb{F}_3) \times (\mathbb{F}_3 \times \mathbb{F}_3) \subseteq D$, $\mathbb{F}_3 \times \mathbb{F}_3 \times \operatorname{End} V^\perp \subseteq D$, $\mathbb{F}_3 \times \mathbb{F}_3 \times D \subseteq D$, & so on...
All of these give contradictions. For example, $Id_{\dim V} D \times Id_{\dim V^\perp} D \subseteq D \, Id_{\dim W}$ & so $(\delta_1 \dots \delta_1, \delta_2 \dots \delta_2) \in (\delta \dots \delta)$ ↯

□