The local theta correspondence (finite field case is similar)

Some nice vector space over F, a local field

Consider reductive dual pairs (G_1, G_2) of subgroups of $Sp(W)$.
There is a canonical double cover of Sp called the metaplectic group denoted by $\widetilde{Sp}$.
By the Stone-Von Neumann theorem, $\widetilde{Sp}$ has a unique irreducible representation (up to scalars)
ω satisfying some nice condition.

(called the Weil representation or metaplectic representation)

The dual pair (G_1, G_2) have commuting images $\widetilde{G}_1$ & $\widetilde{G}_2$ in $\widetilde{Sp}$.
We can then consider the restriction of the Weil representation to $\widetilde{G}_1 \times \widetilde{G}_2$.
Define $A(\pi) = \omega \Big/ \bigcap_{\phi \in \mathrm{Hom}_{\widetilde{G}_1}(\omega, \pi)} \ker \phi$ for an irreducible representation π of $\widetilde{G}_1$

$A(\pi)$ is a smooth representation of $\widetilde{G}_1 \times \widetilde{G}_2$ & decomposes as $A(\pi) = \pi \otimes \Theta(\pi)$
where $\Theta(\pi)$ is a smooth representation of $\widetilde{G}_2$. This is called the <u>theta lift of π</u>.

~~th: $\Theta(\pi)$ is of finite length & if char $F \neq 2$, then it has a unique irreducible quotient~~

Note: ω & Θ depend on a choice of character of $\psi: F \to \mathbb{C}$.

<u>Howe Duality Conjecture</u>: Let (G_1, G_2) be any reductive dual pair & π & π' be an irreducible rep. of $\widetilde{G}_1$. Then,

1) $\Theta(\pi) = 0$ or $\Theta(\pi)$ is irreducible

2) If $\Theta(\pi) = \Theta(\pi') \neq 0$, then $\pi = \pi'$.

- Waldspurger proved this for char $F \neq 2$ in 1990
- Minguez proved for reductive dual pairs of type II in 2008
- (Gan & Takeda) + (Gan & Sun) proved for reductive dual pairs of type I in 2015 which proved the conjecture in full.

Note that (G_1, G_2) was fixed before. Really they arise as groups dependent on some underlying space $(G_1(U), G_2(V))$. If we let V vary in a nice way & take local theta lifts we form <u>Witt towers</u>.

<u>Some natural questions about Witt towers</u>:

1) For U fixed, when is $\Theta(\pi)$ first nonzero? This is called <u>first occurence</u>
there's an ordering in Witt towers

2) Is there a relation between Witt towers?

The answer to (2) is yes! $n_{t_1}(\pi) + n_{t_2}(\pi) = \dim U + d_{D,\epsilon}$ (2015, Sun + Zhu)
First occurence in Witt towers t_1 & t_2; Fixed integer that depends on underlying space

<u>Other connections</u>

(Rallis, 1982) Unramified part of the theta lift can be described by homomorphisms of L-groups for the dual pair $(Sp(W), O(V))$.

The theta lift (1st occurence of a) of a supercuspidal is again supercuspidal. This leads to research about chains of supercuspidals.

Note: Supercuspidals are typically difficult to construct

Ex: If U is the symplectic space of dim. 2 & V_4^- is the anisotropic quadratic space of dimension 4 given by the quaternion norm form, then every supercuspidal rep. $\pi \in Irr_{sc}(Sp(U))$ occurs in the theta correspondence with $O(V_4^-)$.