§6

Def: We say W is <u>alternating</u> if $\langle w,w\rangle = 0$ $\forall w\in W$.

Prop: If W is alternating, then 1) $\varepsilon\tau(\langle w,w'\rangle)+\langle w,w'\rangle=0$ & hence τ is trivial
2) if char $F\neq 2$, then $\varepsilon=-1$ & W is sympletic.

§: 1) $\langle w+w', w+w'\rangle = 0 = \langle w,w\rangle+\langle w,w'\rangle+\langle w',w\rangle+\langle w',w'\rangle = \langle w,w'\rangle+\langle w',w\rangle$
$= \langle w,w'\rangle + \varepsilon\tau(\langle w,w'\rangle)$. ~~So $\varepsilon\tau(\langle w,w'\rangle)=-\langle w,w'\rangle$~~ $\Rightarrow \tau(\langle w,w'\rangle)=-\varepsilon^{-1}\langle w,w'\rangle$
$\Rightarrow \tau(d)=-\varepsilon^{-1}d$
$\tau(e)=e=-\varepsilon^{-1}\cdot e=-\varepsilon^{-1} \Rightarrow \tau(d)=d$.

2) char $F\neq 2$ & τ is trivial $\overset{\text{by}(1)}{\Longrightarrow}$ $\varepsilon\langle w,w'\rangle = -\langle w,w'\rangle \Rightarrow \varepsilon=-1$.
τ trivial $\Rightarrow F'=D=F$ & so W is symplectic $\boxtimes$

(Orthogonalization thm)
th: ~~W is isometric to~~ $W\cong \oplus D(a_i)\oplus W^0$ where W^0 is an alternating space.

Cor: If char $F\neq 2$, any non-sympletic space W is isometric to a orthogonal sum $W\cong\oplus D(a_i)$.

Note: These decompositions are not unique (consider a quadratic space). However, it allows us to define invariants.

§: Classical procedure of Schmidt orthogonalization:
If W is not alternating, let $w\in W \ni a=\langle w,w\rangle\neq 0$. Then $a=\varepsilon\tau(a)$. Complete W to a basis $\{w,v_2,\dots,v_n\}$ of W. Choose $d\in D \ni wd+v_2$ is orthogonal to w, ..., etc.
Thus, we can assume each v_i is orthogonal to w. The v_i's generate an ε-Hermitian space W' of dim. $n-1$. Also $W\cong\langle w\rangle\oplus W'$. If W' is not alternating, iterate/continue the above process to get $W\cong \oplus\langle w_i\rangle\oplus W^0$ where W^0 is alternating & orthogonal to $\oplus\langle w_i\rangle$. $\langle w_i\rangle$ is 1-dim'l & hence $\langle w_i\rangle\cong D(a_i)$ for some a_i $\boxtimes$

Invariants

f: By the orthogonalization thm, $W \cong \bigoplus D(a_i) \oplus W^0$.

1) If W is quadratic, the determinant is $d(W) = \prod a_i$. Note $d(W) \in F^\times/(F^\times)^2$.

2) If F is local or global, the Hasse invariant is $h(W) = \prod_{i<j} (a_i, a_j)$ where $(\cdot,\cdot)$ is the Hilbert symbol.

3) If $F = \mathbb{R}$, the signature is $s(W) = p - q$ where p is the number of positive a_i's & q is the number of negative a_i's

Note: The dimension & signature determine p & q & the converse is also true. So this well-defined.
($\dim = p+q$ signature $= p-q$)

ark We can generalize the determinant & signature.
Let $N: D \to F'$ be reduced norm & $N_{F'/F}: F' \to F$ be the norm. Then,
the determinant is

$$d(W) = \begin{cases} \text{the image of } N(\prod a_i) \text{ in } F^\times/(F^\times)^2 & \text{if } \tau \text{ is of the } 1^{st} \text{ Kind} \\ \text{the image of } N(\prod a_i) \text{ in } F^\times / N_{F'/F}(F^\times) & \text{if } \tau \text{ is of the } 2^{nd} \text{ Kind.} \end{cases}$$

The signature $s(W)$ generalizes to Hermitian spaces on $\mathbb{C}$ (similar def'n)

These invariants, along w/dimension, are sufficient to classify the Hermitian spaces if F is finite or local, up to one exception (Anti-Hermitian spaces on the Quaternions w/canonical involution)

§7

th: If W is alternating, then $W \cong m\overset{\text{hyperbolic plane}}{H}$ & $\overset{\dim W}{n} = 2m$, where $mH = \overbrace{H \oplus H \oplus \cdots \oplus H}^{m \text{ times}}$

Pf: If $w \neq 0$, $\exists v \in W \ni d = \langle w, v \rangle \neq 0$ since $\langle \cdot, \cdot \rangle$ is nondegenerate. Let $w' := v d^{-1}$.
Let W_1 be the submodule of W generated by w & w'. Note W_1 is isometric to H.
Let W_2 be the orthogonal complement of W_1 in W. Since W_1 is non-degenerate, $W = W_1 \oplus W_2$. Now, W_2 is alternating & has dimension $n-2$. Repeat. $\boxtimes$

Cor: 1) In general, $W \cong \oplus D(a_i) \oplus mH$.
2) The alternating spaces are classified by their dimension $n \in 2 \cdot \mathbb{N}$.
3) $D(a) \oplus D(-a) \cong H$ since $D(a) \oplus D(-a)$ is alternating of dim. 2
4) Let $-W$ be the ϵ-Hermitian space with product $-\langle \cdot, \cdot \rangle$. Then, $W + (-W) \cong nH$.

Def: A space isometric to nH is called <u>hyperbolic</u>.

§8

(hyperbolic basis) A similar method to §7 gives the following result

Prop: Let $V \subseteq W$ be a right D-submodule $\ni \langle v, v' \rangle = 0$ $\forall v, v' \in V$. Then, $\forall$ basis $\{e_i\}$ of V $\exists \{f_i\} \in W \ni \langle e_i, f_j \rangle = \delta_{i,j}$ & the Hermitian product is null on the space V^* generated by the $\{f_i\}$. ([illegible])

Def: If $r = \dim V$, ~~[illegible]~~ then $V \oplus V^*$ is ε-Hermitian & $V \oplus V^* \cong rH$. The basis $\{e_i, f_i\}$ of $V \oplus V^*$ is called a <u>hyperbolic basis of $V \oplus V^*$</u>. We say that V is <u>totally isotropic</u> if it satisfies the above conditions.

Def: If $\exists w \in W$, $w \neq 0$ & $\langle w, w \rangle = 0$, we say w is <u>isotropic</u>

Prop: If w is isotropic, then $\langle w \rangle \cong H$ & hence W contains a subspace isometric to H.

Def: If W has no isotropic elements, we say W is <u>anisotropic</u>.

~~[illegible]: W is isometric to the orthogonal sum $W \cong mH \oplus W^0$ where W^0 is anisotropic.~~

~~Pf: Let $V = \{w \in W \mid \langle w, w \rangle = 0\}$. Then V is alternating & so $V \cong mH$. Then $W^0 := V^\perp$ [illegible] is anisotropic & $W \cong V \oplus W^0 \cong mH \oplus W^0$.~~ $\boxtimes$

~~Note [illegible] $\langle v, v \rangle$ [illegible]~~

~~Pf: By orthogonalization thm $W \cong W' \oplus (\oplus D(a_i))$ where W' is alternating. By prev. thm $W' \cong mH$. Let $W^0 = \oplus D(a_i)$. It remains to to show that W^0 is anisotropic. Let $\partial \in D(a_i)$. Then, $\langle \partial, \partial \rangle = \tau(\partial)$ The a_i were chosen in the pf $\ni a_i \neq 0$ & hence $\langle \partial, \partial \rangle = \tau(\partial) a_i \partial \neq 0$ $\forall \partial \in D(a_i)$, $\partial \neq 0$. So, W^0 is anisotropic.~~ $\boxtimes$

~~($a_i \neq 0$, $\partial \neq 0$, $\tau(\partial) \neq 0$ since $\tau^{0}(\tau(\partial)) = \partial$ & $\tau(0) = 0$)~~