Parabolics

Extension of Scalars

Let $\bar{F}$ be an algebraic closure of F & L be a field $\ni F \subseteq L \subseteq \bar{F}$.
Let W be an ε-Hermitian space on (D, τ). The group $U(W)$ is the group of rational points on F of an algebraic group $\mathcal{U}$.
We have $\mathcal{U}(L) = U(W_L)$ where $W_L = W \otimes_F L$ is the right $D_L := D \otimes_F L$-module equipped with a product extending that of W.
Alternatively, if i is the involution of $A = \mathrm{End}_D W \ni U(W) = \{u \in A \mid i(u) \cdot u = I\}$, then $\mathcal{U}(L) = \{a \in A_L \mid i(a)a = I\}$ where $A_L = \mathrm{End}_{D_L} W_L$.

Def: We define the group $S\mathcal{U}$ to be the kernel of the determinant in $\mathcal{U}$.

Notation: Let $O(n)$, $Sp(2m)$, & $GL(n)$ be the three unitary groups on $\bar{F}$.
Also, we let F be a finite, local, or global field.

Lemma: The group $\mathcal{U}(\bar{F})$ is
1) $GL(rn)$ if a) τ is of the second ~~type~~ kind with $\begin{cases} r=1 \text{ if } D=F \\ r=2 \text{ if } D \text{ is quaternionic} \end{cases}$

b) W is of type 2 with $r^2 = [D:F]$.

2) $O(rn)$ if τ is of the first kind with $\begin{cases} \varepsilon=1, r=1 \text{ if } D=F \\ \varepsilon=-1, r=2 \text{ if } D \text{ is quaternionic} \end{cases}$

3) $Sp(rn)$ if τ is of the first kind with $\begin{cases} \varepsilon=-1, r=1 \text{ if } D=F \\ \varepsilon=1, r=2 \text{ if } D \text{ is quaternionic.} \end{cases}$

Cor: $\mathcal{U}$ is a Zariski-connected reductive group, except if W is orthogonal or Anti-Hermitian $(\varepsilon=-1)$ on a set of quaternions with canonical involution. In this case, $S\mathcal{U}$ is a connected reductive group.
see Humphrey's → (PF: GL & Sp are connected over $\bar{F}$, but O is not. However SO is connected over $\bar{F}$. All are reductive).

Cor: If W is orthogonal, then $SU(W) \subseteq U(W)$ is of index 2 & $U(W)$ is not the group of F-rational points of a connective reductive group.
If W is ε-Hermitian on a set of quaternions equipped with canonical involution, then $SU(W) = U(W)$ (but $S\mathcal{U} \neq \mathcal{U}$!) is the group of F-rational points of a semisimple connected group.

PF: The 1st claim is immediate. If D is quaternionic, then the Dieudonné determinant $A^\times \to D^\times/(D^\times, D^\times)$ is trivial on $U(W)$. So $\overset{SU(W).}{\overset{\shortparallel}{U(W)}}$ □

dea of
of lemma

- If τ is of the 2^{nd} kind, then $A_{\bar{F}} \cong M(rn,\bar{F}) \times M(rn,\bar{F})$ equipped with an involution i that permutes the two factors. In this case, $U(\bar{F}) = GL(rn,\bar{F})$.
- If W is of type 2, the result is clear.
- If $W = \oplus F(a_i)$ or $\oplus_m H$ is orthogonal or symplectic, respectively, $(D=F)$ then $A = M(n,F)$ equipped with involution $a \mapsto h^t a h^{-1}$ where $h = \mathrm{diag}(a_i)$ if $W = \oplus F(a_i)$ or $h = \mathrm{diag}(u)$ where $u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if $W = \oplus mH$.
  We find $A_{\bar{F}} = M(n,\bar{F})$ with similar involution & hence $U(\bar{F})$ is $O(n)$ or $\underset{\substack{\| \\ Sp(2m)}}{Sp(n)}$.
- If D is quaternionic with canonical involution, then $A_{\bar{F}} = M(2n,\bar{F})$ is equipped with involution $a \mapsto h\,\mathrm{diag}(u)^t a\,(h\,\mathrm{diag}(u))^{-1}$ & hu is $-\epsilon$-symmetric if h is ϵ-symmetric. So $U(\bar{F})$ is $O(2n)$ or $Sp(2n)$. $\square$

Def: A *flag* is a set of submodules $\{0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_t\}$ of W.
We say a flag is totally isotropic if each V_i is totally isotropic in W.

~~A totally isotropic flag is acted upon naturally~~

The set of totally isotropic flags in acted on naturally by U. Let $\Phi = \{0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_t\}$ be a totally isotropic flag.

(Orbits under U(W)) By Witt's theorem, the only invariant is $\{n_1 \leq \cdots \leq n_t\}$ where $n_i = \dim_D X_i$.
Thus, the orbit of Φ is $\{\Psi \text{ totally isotropic flag} \mid \Psi = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_t\} \ \& \ \dim_D V_i = \dim_D X_i \ \forall i\}$.

(Orbits under SO) If W is orthogonal, then an O-orbit is an SO-orbit, except in the exceptional case: $W \cong mH$ hyperbolic & $n_t = \dim X_t = m$. In the exceptional case, an O-orbit is a union of 2 SO-orbits.

Thus, Φ is not SO-conjugate to the totally isotropic flag $\Phi' = \{0 \subsetneq X_1' \subsetneq \cdots \subsetneq X_t'\}$ where $X_i' = X_i$ if $i < t$ & X_t' is generated by by $\{e_i \mid 1 \leq i \leq m-1\} \cup \{f_m\}$ where $\{e_i, f_i\}_{i=1}^m$ is a hyperbolic basis of $W \ni \{e_i \mid 1 \leq i \leq m\}$ is a basis for X_t.

Def: We call the stabilizer in U (resp. SU) of a totally isotropic flag Φ of W a <u>parabolic subgroup</u> of W, which we denote by $P(\Phi)$ (resp. $P^+(\Phi)$).

Prop: By previous corollary ($[SO:O]=2$), $P^+(\Phi) \subseteq P(\Phi)$ has index 2 except in the exceptional case where $P^+(\Phi) = P(\Phi)$.

<u>The very exceptional case:</u> $W \cong mH$ orthogonal hyperbolic with $\dim X_t = m$ or $m-1$.
Let Φ' be the totally isotropic flag constructed in (orbits under SO). Let Φ'' be the totally isotropic flag obtained by removing X_t from Φ.
Then, $P^+(\Phi) = P^+(\Phi') = P^+(\Phi'')$ but only $P(\Phi) \neq P(\Phi')$.

Prop: If $P^+(\Phi) = P^+(\Phi_1)$, then $\Phi = \Phi_1$, except in the very exceptional case where $\Phi_1 \in \{\Phi, \Phi', \Phi''\}$.

We will prove this later. We make some remarks & corollaries first.

Remark: If $P(\Phi) = P(\Phi_1)$ then $\Phi = \Phi_1$, except in the very exceptional case where $\Phi_1 \in \{\Phi, \Phi'\}$.

Normalizers)
Cor: $P(\Phi)$ is its own normalizer in U except in the very exceptional case, where $P(\Phi)$ is of index 2 in its normalizer. $P^+(\Phi)$ is equal to its normalizer in SU in all cases.

onjugation classes)
Cor: $P(\Phi)$ is conjugate in U to $P(\Phi_1)$ iff Φ & Φ_1 are in the same U-orbit.
The only invariant is $\{\dim X_1 \leq \dots \leq \dim X_t\}$. In the exceptional case, $\Phi' = u\Phi$ where $u \in O$ (<u>not</u> SO).

) $P^+(\Phi)$ is conjugate in SU to $P^+(\Phi_1)$ iff Φ & Φ_1 are in the same U-orbit in the nonexceptional cases. The only invariant is $\{n_1 \leq \dots \leq n_r\}$. In the very exceptional case, Φ & Φ'' are not in the same U-orbit. In the exceptional case (not very exceptional) $P^+(\Phi)$ is not SU-conjugate to $P^+(\Phi')$.

The <u>maximal parabolics</u> are $P(X) = P(0 \subsetneq X)$, up to conjugation there are m classes where is the Witt index of W, & these are classified by $\dim_D X$.

The maximal parabolics of SO are $P^+(X)$ except in the exceptional case where we have $\dim X \neq m-1$ ($P(X) \cap P(X') = P(X \cap X')$ if X is Lagrangian). Up to conjugation, there are m classes & these are classified by $\dim_D X$, except in the exceptional case where there is 1 class for each dimension $< m-1$, none for $m-1$, & 2 for m.

Levi sbgp
$\downarrow$

There is a split exact sequence $1\to N(X)\to P(X)\to M(X)\to 1$ where $N(X)$ is the unipotent radical of $P(X)$.

itt
ecomp
Lagrangian
of mH

If $W = mH \oplus W^0 = (X+X^*)\oplus W^0$, then $M(X)\cong GL_D(X)\times U(W^0)$ & we have an exact sequence $1\to S^2(X,-\varepsilon)\to N(X)\to \mathrm{Hom}_D(W^0,X)\to 1$.
The extension is central.

Def: Suppose there are nondegenerate couplings on $X\times X^*$ & on $Y\times Y^*$ denoted by $\langle\cdot,\cdot\rangle_X$ & $\langle\cdot,\cdot\rangle_Y$. If $f\in \mathrm{Hom}_D(X,Y)$, define the adjoint function $f^*\in\mathrm{Hom}_D(Y^*,X^*)$ by $\langle f(x),y^*\rangle_Y=\langle x,f^*(y^*)\rangle_X$.

The Levi subgroup $M(X)$ of $P(X)$ associated to the decomposition $W=X+W^0+X^*$ is

$$M(X)=\{m(g,u):=\mathrm{diag}(g,u,(g^*)^{-1})\mid g\in GL_D(X),\ u\in U(W^0)\}.$$

ef: Let $N_1(X)=\left\{n_1(s):=\begin{bmatrix}1&0&s\\ &1&0\\ 0&0&1\end{bmatrix}\ \middle|\ s\in\mathrm{Hom}_D(X^*,X),\ s^*=-s\right\}$.

$N_1(X)$ is a sbgp of $N(X)$. Furthermore $N_1(X)$ identifies with $S^2(X^*,-\varepsilon)$.

ef: Let $N_2(X):=\left\{n_2(h):=\begin{bmatrix}1&h&-\frac{hh^*}{2}\\ 0&1&-h^*\\ 0&0&1\end{bmatrix}\ \middle|\ h\in\mathrm{Hom}(W^0,X)\right\}$.

Then $N_2(X)$ is a subgroup of $N(X)$ & any $n\in N(X)$ can be written uniquely as $n=\overset{N_1(X)}{\overset{\cup}{n_1(s)}}\,\overset{N_2(X)}{\overset{\cup}{n_2(h)}}$.

p: 1) $m(g,u)n_1(s)m(g,u)^{-1}=n_1(gsg^*)$

2) $m(g,u)n_2(h)m(g,u)^{-1}=n_2(ghu^{-1})$

(1)+(2) show that $M(X)$ acts naturally on $N(X)$

3) $n_2(h)n_2(k)=n_2(h+k)\,n_1\left(\frac{-hk^*+kh^*}{2}\right)$

4) The commutator of two element of $N_2(X)$ is $[n_2(h),n_2(k)]=n_1(-hk^*+kh^*)$

a 1) If $W^0\neq\{0\}$, then the group of commutators of $N(X)$ is $N_1(X)$.

2) $N(X)$ is Abelian iff

a) $W^0=\{0\}$ & thus $N(X)=N_1(X)$ or

b) W^0 is orthogonal & $\dim_D X=1$. So, $N(X)=N_2(X)$.