Restriction of the Weil Rep. (following Kudla's notes)

The results for the theta correspondence have only been discussed for $\tilde{H}_1, \tilde{H}_2$.
We now aim to discuss H_1, H_2.

For simplicity, assume $H_1 = O(V)$ where $\dim_F V = m$ &
$H_2 = Sp(W)$ where $\dim_F W = n$.
Let $\mathbb{W} = V \otimes_F W$. Then, there is an inclusion $j: H_1 \times H_2 \to Sp(\mathbb{W})$.
Let $j_V = j\restriction_{1 \times Sp(W)}$ & $j_W = j\restriction_{O(V) \times 1}$.

Prop: The homomorphism $j_V: Sp(W) \to Sp(\mathbb{W})$ lifts uniquely to a homomorphism $\tilde{j}_V: \tilde{Sp}(W) \to \tilde{Sp}(\mathbb{W})$ whose restriction to $\mathbb{C}^\times$ is $z \mapsto z^m$.

In particular, $\tilde{j}_V\restriction_{\widehat{Sp(W)}}$ ($\widehat{Sp(W)}$ = 2-fold cover of $Sp(W)$) factors through $Sp(W)$ iff $m = \dim_F V$ is even.

(Won't prove; see cor 3.3 in Kudla's notes).

Remark: Except in this case ($\dim_F V = $ odd), extensions $\tilde{H}_1, \tilde{H}_2$ of irred. dual pairs of (H_1, H_2) in $Sp(\mathbb{W})$ are always split.

Consider the pullback $\tilde{j}_V^*(\underbrace{\omega_\psi}_{\text{Weil rep. of } \widetilde{Sp(\mathbb{W})}})$. This is a rep. of $\widetilde{Sp(W)}$.
Its restriction to $\mathbb{C}^\times$ is $\tilde{j}_V^*(\omega_\psi)(z) = z^m \cdot id_S$ (S = Schrödinger model for ω_ψ).
Its inconvenient to have this central character depend on $m = \dim_F V$ & so a modification is made.

$\widetilde{Sp(W)}$ has a unique character λ whose restriction to $\mathbb{C}^\times$ is $\lambda(z)=z^2$ & $\operatorname{Ker}\lambda = \widehat{Sp(W)}$.

Let $A_m(\widetilde{Sp(W)})$ be smooth rep.'s whose restriction to $\mathbb{C}^\times$ is $z \mapsto z^m$.
If $m-m'$ is even, then $A_m(\widetilde{Sp(W)})$ naturally identifies with $A_{m'}(\widetilde{Sp(W)})$ via the tensor product with $\lambda^{\pm\frac{m-m'}{2}}$.

Thus, it is enough to consider $A_0(\widetilde{Sp(W)}) \cong A(Sp(W))$, the category of rep's of $\widetilde{Sp(W)}$ that factor through $Sp(W)$ &

$$A(\widetilde{Sp(W)})^{gen} := A_1(\widetilde{Sp(W)}) \text{ the category of } \underline{\text{genuine}} \text{ rep's of } \widetilde{Sp(W)}.$$

We define the tensor products & contragradients in $A(\widetilde{Sp(W)})^{gen}$ by
$\pi_1 \otimes^{gen} \pi_2 = \pi_1 \otimes \pi_2 \otimes \lambda^{-2} \in A(Sp(W))$ where $\pi_1, \pi_2 \in A(\widetilde{Sp(W)})^{gen}$ &
$\widetilde{\pi}^{gen} = \widetilde{\pi} \otimes \lambda^2$ where $\widetilde{\pi} \in A_{-1}(\widetilde{Sp(W)})$ is the usual contragradient of $\pi \in A(\widetilde{Sp(W)})^{gen}$.

The $\underline{\text{Weil rep.}}$ $\underset{w_{V,\psi}}{\vee}$ $\underline{\text{associated to } \psi \text{ \& } V}$ is the image of $j_V^*(w_\psi)$ in $\begin{cases} A(Sp(W)) & \text{if } m=\dim_F V \text{ is even} \\ A(\widetilde{Sp(W)})^{gen} & \text{if } m=\dim_F V \text{ is odd}. \end{cases}$

That is,
$$w_{V,\psi} = \begin{cases} \lambda^{\frac{-m}{2}} \otimes j_V^*(w_\psi) & \text{if } m \text{ is even} \\ \lambda^{\frac{-m-1}{2}} \otimes j_V^*(w_\psi) & \text{if } m \text{ is odd}. \end{cases}$$

ef: As before ~~Repeating for $w_{V,\psi}$~~ (replacing w_ψ's by $w_{V,\psi}$'s) we obtain the $\underline{\text{theta correspondence}}$ between $\begin{cases} O(V) \text{ \& } Sp(W) & \text{if } \dim_F V = \text{even} \\ O(V) \text{ \& } \widetilde{Sp(W)} & \text{if } \dim_F V = \text{odd}. \end{cases}$

For $\dim_F V$ even & π an irred. admissible rep. of $O(V)$, $\Theta(\pi)$ is an irred. (if nonzero) smooth rep. of $Sp(W)$.

For $\dim_F V$ odd & π an irred. admissible of $O(V)$, if $\Theta(\pi)$ is nonzero, then it is an irred. smooth rep. of $\widetilde{Sp(W)}$.

An Example

Let $V=F$ be a 1 dimensional quadratic space with quadratic form $a\cdot x^2$ where $a\in F^\times$. ~~Let~~

~~$\chi_V(x) = (x,a)_F = \begin{cases} 1 & \text{if } z^2 = ax^2 + by^2 \text{ has a nonzero solution } (x,y,z)\in F^3 \\ -1 & \text{o.w.} \end{cases}$~~

~~$\begin{cases} 1 & \text{if } z^2 = xy^2 + aw^2 \text{ has a nonzero solution } (z,y,w)\in F^3 \\ -1 & \text{o.w.} \end{cases}$~~

We have $\omega_{\psi_a} = \omega_{\psi_a}^{+} \oplus \omega_{\psi_a}^{-}$ where $\omega_{\psi_a}^{\pm}$ is irred.

Consider the dual pair $(\widetilde{Sp}(W_n), O(V))$. ~~We have $\omega_{\psi_a} = \omega_{\psi_a}^{+} \oplus \omega_{\psi_a}^{-}$ where $\omega_{\psi_a}^{\pm}$ is irred.~~

Also,

$$\Theta(1_V) = \omega_{\psi_a}^{+} \;\&\; \Theta(sgn_V) = \omega_{\psi_a}^{-}$$ where ω_{ψ_a} is the Weil rep of $\widetilde{Sp}(W_n)$ & $\omega_{\psi_a}^{\pm}$ are the even & odd parts.

There is a conservation relation $n(1_V) + n(sgn_V) = 1$. By convenction, $n(1_V) = 0$ (since $\Theta(1_V) = 1_W$ for the 0-dim'l space W_0). So, $n(sgn_V) = 1$. Thus, the rep. $\omega_{\psi_a}^{-}$ of $\widetilde{Sp}(W_1)$ is the 1st occurence of sgn_V. sgn_V is supercuspidal & therefore $\omega_{\psi_a}^{-}$ is a supercuspidal rep. of $\widetilde{Sp}(W_1)$.