I, §20

Tensor product Decompositions of Symplectic Space

Let $(W,\langle\cdot,\cdot\rangle)$ be a symplectic space of dimension $2n$ over F. Our classification shows that irred. dual pairs of $Sp(W)(=U(W))$ correspond to ε-Hermitian tensor product decompositions of W. (or Lagrangian subspaces which are the subject of later study)

Let $t_{D/F}\in \mathrm{Hom}_F(D,F)$ be ∋ the bilinear form $(d,d'):=t_{D/F}(dd')$ is non-degenerate ($\forall d,d'\in D$.)
In general, $t_{D/F}$ is the reduced trace.

Lemma: If $(W_1,\langle\cdot,\cdot\rangle_1)$ & $(W_2,\langle\cdot,\cdot\rangle_2)$ are two ε_i-Hermitian D-modules on the right & left, respectively, ∋ $-1=\varepsilon_1\varepsilon_2$, then the tensor product $W=W_1\otimes_D W_2$ equipped with the form

$$\langle\langle w_1\otimes w_2, w_1'\otimes w_2'\rangle\rangle := t_{D/F}(\langle w_1,w_1'\rangle_1\, \tau(\langle w_2,w_2'\rangle_2)) \quad \text{for } w_i,w_i'\in W_i$$

is symplectic. Inversely, every ε-Hermitian tensor product decomp. of W is of this type.

PF: W is verified to be symplectic directly. We proceed to the 2^{nd} part.
Let $W=W_1\otimes_D W_2$ be an ε-Hermitian tensor product decomp. of $(W,\langle\cdot,\cdot\rangle)$. The form $\langle\langle\cdot,\cdot\rangle\rangle$ induces an involution on $\mathrm{End}_F W$ that coincides with the ~~involution~~ adjoint involution associated to $\langle\cdot,\cdot\rangle_i$ on $\mathrm{End}_D W_i$. Two involutions on $\mathrm{End}_F W$ differ by an interior automorphism. Any interior automorphism of $\mathrm{End}_F W$ that is trivial on $\mathrm{End}_D W_i$, $i=1,2$, is given by conjugation by a nonzero element in F' (center of D). Thus, modifying $\langle\cdot,\cdot\rangle_i$ appropriately, we obtain $\langle\langle\cdot,\cdot\rangle\rangle=\langle\cdot,\cdot\rangle$. □

OK

Restriction of Scalars

1) $\forall$ Anti-Hermitian space $(W, \langle\cdot,\cdot\rangle)$ on $(D, \overset{\text{involution}}{\tau})$ & any homomorphism $t_{D/F} \in Hom(D,F) \Rightarrow$ $(x,y) \mapsto t_{D/F}(xy)$ is a nondegenerate bilinear form $D \times D \to F$ (in general, the trace), the space $(W, t_{D/F}\langle\cdot,\cdot\rangle)$ is symplectic on F & we call it the space generated by <u>restriction of scalars</u> of $(W, \langle\cdot,\cdot\rangle)$ & $t_{D/F}$.

2) A dual pair in $Sp(W)$ gives a dual pair in $Sp(W')$ if W' is a restriction of scalars of W except if the pair is trivial: $(\{\pm 1\}, Sp(W))$.

- List of Irred. Dual pairs of $Sp(2n, F)$ not arising as restriction of scalars of $Sp(2n', F')$ where $n'[F:F'] = n$

a) Pairs of type 2: $(GL(m,D), GL(m',D))$ where D has center F, $[D:F] = d$, & $n = mm'd$.
division algebra

b) Pairs of type 1:
- $(O(m,F), Sp(2m',F))$ where $O(m,F) \neq O(2, F_3)$ & $n = mm'$
- $(U^+(m,D), U^-(m,D))$ where D/F is a quadratic extension or quaternions with canonical involution & $U^{\pm}(m,D)$ is the unitary group of a $\pm$Hermitian form on m variables on D. Note $m' \neq 1$ if D is a quaternion algebra. Also, $mm'd = 2n$.

Remark: It is possible to classify tensor products of W in general & determine the classification of $U(W)$ for finite & local fields without too much difficulty. However, this is not used in the Howe correspondence. It is only mildly more complicated as the Witt group may be nontrivial (however, one can still define a restriction of scalars).