Local Theta Correspondence

Let W be a symplectic v.s. over F, $\psi: F \to \mathbb{C}^\times$ nontrivial character. Let ω_ψ be the Weil representation of $\widetilde{Sp}(W)$.

Lemma: Let (H_1, H_2) be a reductive dual pair of $Sp(W)$ & $\widetilde{H}_1, \widetilde{H}_2$ be their inverse images in $\widetilde{Sp}(W)$. Then, $\widetilde{H}_1 = C_{\widetilde{Sp}(W)}(\widetilde{H}_2)$ & $\widetilde{H}_2 = C_{\widetilde{Sp}(W)}(\widetilde{H}_1)$.

Pf: Let M_g be the intertwining operator $M_g \rho_\psi = \rho_\psi^g M_g$ where ρ_ψ is the rep. of $H(W)$ defined previously.

Lemma II.5 of $\boxed{\text{MVW}}$ says if $g_1, g_2 \in Sp(W) \ni g_1 g_2 = g_2 g_1$, then $M_{g_1} M_{g_2} = M_{g_2} M_{g_1}$. Since $H_1 = C_{Sp(W)}(H_2)$ & $H_2 = C_{Sp(W)}(H_1)$, the claim follows. $\square$

Note: 1) If F is finite, H_1 & H_2 are split because $Sp(W) \cong \widetilde{Sp}(W)$ is

2) If F is local nonArchimedean, we have 2 cases.

Case 1: Suppose $W = W_1 \otimes_D W_2$ where W_i are ε_i-Hermitian (type 1) & $H_i = U(W_i)$. Then H_1 is split except if D is commutative w/ trivial involution & $\varepsilon_1 = -1$ (i.e. W_1 is symplectic) & $\dim_D W_2$ is odd. In this case $\widetilde{H}_1 \cong \widetilde{Sp}(W_1)$.

Case 2: Suppose $W = X_1 \otimes_D X_2 + (X_1 \otimes_D X_2)^*$ & $H_i = GL_D(X_i)$. Then H_1 & H_2 are split in $\widetilde{Sp}(W)$. Fixing a suitable splitting, we obtain via the Schrödinger model that ω_ψ restricted to $H_1 \times H_2$ is the rep. of $H_1 \times H_2$ on the Schwartz space $S(X_1 \otimes_D X_2)$ defined by

$$\omega_\psi(h_1, h_2) f(x_1 \otimes x_2) = |\det h_1|_F^{-\frac{m_2}{2}} \cdot |\det h_2|_F^{-\frac{m_1}{2}} f(h_1^{-1} x_1 \otimes h_2^{-1} x_2).$$

where $m_i = \dim_D X_i$

Howe's Conjecture

Let G be a closed sbgp of $Sp(W)$. Let $i: \mathbb{C}^\times \hookrightarrow \tilde{G}$. In the following, assume (π, V) is a rep. of $\tilde{G}$ $\ni$ $\pi \circ i(z) = z \cdot Id_V$ $\forall z \in \mathbb{C}^\times$.

Let (ω_ψ, S) be the metaplectic rep. of $\widetilde{Sp}(W)$ & $R_\psi(\tilde{G})$ be the set $\overset{\text{isomorphism classes}}{\text{of}}$ irred. admissible rep.'s (π, V) of $\tilde{G}$ $\ni$ $Hom_{\tilde{G}}(S, V) \neq \{0\}$.

Let (H_1, H_2) be a reductive dual pair of $Sp(W)$ & $\pi \in R_\psi(\widetilde{H_1 \times H_2})$. Then, there exists irred. admissible rep.'s π_1, π_2 of $\tilde{H}_1$ & $\tilde{H}_2$, respectively, unique up to isomorphism $\ni$ π can be obtained by factoring the rep. $\pi_1 \otimes \pi_2$ of $\tilde{H}_1 \times \tilde{H}_2$ by the projection $\tilde{H}_1 \times \tilde{H}_2 \twoheadrightarrow \widetilde{H_1 \times H_2}$ (Flath, 1979).

We abuse notation & write $\pi = \pi_1 \otimes \pi_2$
Since $\pi \in R_\psi(\widetilde{H_1 \times H_2})$ is a quotient of ω_ψ, π_1 & π_2 are also quotients of ω_ψ. Thus $\pi_i \in R_\psi(\tilde{H}_i)$. That $R_\psi(\widetilde{H_1 \times H_2})$ can be identified as a subset of $R_\psi(\tilde{H}_1) \times R_\psi(\tilde{H}_2)$.

towe) Conjecture: If F is local, non Archimedean, then $R_\psi(\widetilde{H_1 \times H_2})$ is the graph of a bijection between $R_\psi(\tilde{H}_1)$ & $R_\psi(\tilde{H}_2)$.

marks: 1) This has been proven (Waldspurger, residue characteristic $\neq 2$)
(Minguez; Gan, Takeda; Gan, Sun)

2) Howe proved this if $F = \mathbb{R}$ (Transcending classical invariant theory)

3) A stronger conjecture was actually proven which we work towards next.

Some Lemmas

Lemma: Let G_1, G_2 be locally compact, totally disconnected groups. Let (π_1, V_1) be an irred. admissible rep. of G_1 & (π_2, V_2) be a smooth rep. of G_2. Let V be a $G_1 \times G_2$-invariant subspace of $V_1 \otimes V_2$. Then, $\exists$ a subspace $V_2' \subseteq V_2$ invariant under $G_2 \ni V = V_1 \otimes V_2'$.

PF: Set $V_2' := \{v_2 \in V_2 \mid \forall v_1 \in V_1,\ v_1 \otimes v_2 \in V\}$. V_2' is G_2-stable & $V_1 \otimes V_2' \subseteq V$.
Thus, quotienting V by $V_1 \otimes V_2'$ reduces us to the case where $V_2' = \{0\}$. Thus, assume WLOG $V_2' = \{0\}$. We WTS $V = \{0\}$.

Assume $V \neq \{0\}$ for contradiction. Let $v \in V \setminus \{0\}$. Write $v = \sum_{i=1}^{n} v_1^i \otimes v_2^i$ where v_1^i are linear independent & $v_2^i \neq 0\ \forall i$. π_1 is admissible. So, $\exists K_i$ open compact sbgp of $G \ni v_1^i \in V_1^{K_i} \left(= \{v \in V_1 \mid \pi(k)v_1 = v_1,\ \forall k \in K_i\}\right)$. Set $K = \bigcap_{i=1}^{n} K_i$. Then K is an open sbgp of G & $v_1^i \in V_1^K\ \forall i$.
Let $\mathcal{H}_K$ be the algebra (w/convolution) of bi-K-invariant distributions of compact support on G_1 ~~the Hecke algebra~~. The induced representation of π_1 of $\mathcal{H}_K$ in V_1^K is irreducible (Bernstein-Zelevinsky).
Also, $\dim V_1^K < \infty$ (π_1 is admissible). $\left(\pi_1(f)v = \int_{G_1} f(g)\pi(g)v\,dg,\ f \in \mathcal{H}_K,\ v \in V_1^K\right)$
Thus, the map $\pi_1 : \mathcal{H}_K \twoheadrightarrow \mathrm{End}_{\mathbb{C}} V_1^K$ is surjective
& $\exists f \in \mathcal{H}_K \ni \pi_1(f)v_1^i = \begin{cases} 0 & \text{if } i \neq 1 \\ v_1^i & \text{if } i = 1 \end{cases}$. So, $v_1^1 \otimes v_2^1 = \pi_1(f)v \in V$.

arbitrary!

Let $v_1 \in V_1$. By irred. of π_1, $\exists$ a distribution f' of compact support on $G_1 \ni$ $\pi_1(f')v_1^1 = v_1$. So, $v_1 \otimes v_2^1 = \pi_1(f')(v_1^1 \otimes v_2^1) \in V$. Hence $0 \neq v_2^1 \in V_2'$ ↯ $\square$

(not allways 0 by above arg. $(\pi(f)(v_1^1 \otimes v_2^1) = v_1^1 \otimes v_2^1 \neq 0)$)

Let G_1, G_2 be locally compact, totally disconnected groups, (π_1, V_1) an irred. admissible rep. of G_1, & (π, V) a smooth rep. of $G_1 \times G_2$. Suppose $\bigcap_{f \in \mathrm{Hom}_{G_1}(V, V_1)} \ker f = 0$.

Then, up to isomorphism, there is a unique smooth rep. (π_2, V_2) of G_2 ∋ $\pi \cong \pi_1 \otimes \pi_2$.

∀ G_1-module U_1, let $U_1[G_1]$ be its largest quotient on which G_1 acts trivially.
Let $(\check{\pi}_1, \check{V}_1)$ be the contragradient of (π_1, V_1). Since π_1 is irred., $[\check{V}_1 \otimes V_1][G_1] \cong \mathbb{C}$.

is) Suppose that π_2 exists. Then, $(\check{V}_1 \otimes V)[G_1] \cong (\check{V}_1 \otimes V_1 \otimes V_2)[G_1] \cong (\check{V}_1 \otimes V_1)[G_1] \otimes V_2 \cong V_2$.
Thus, V_2 is unique up to isomorphism.

Set $V_2' = (\check{V}_1 \otimes V)[G_1]$ & let $p: \check{V}_1 \otimes V \to V_2'$ be the natural projection.
Then V_2' naturally has a smooth π_2' of G_2. Define a linear operator $\varphi: V \to \mathrm{Hom}_{\mathbb{C}}(\check{V}_1, V_2')$ by $v \mapsto (\check{v}_1 \mapsto p(\check{v}_1 \otimes v))$. φ intertwines π with the action of $G_1 \times G_2$ on $\mathrm{Hom}_{\mathbb{C}}(\check{V}_1, V_2')$ induced from $\check{\pi}_1$ & π_2'.

Let $v \in V$ & K open sbgp of G_1 fixing v, e_K the associated idempotent of $S_c^*(G_1)$. (distributions of compact support)
For $\check{v}_1 \in \check{V}_1$, we have

$$\varphi(v)(\check{v}_1) = p(\check{v}_1 \otimes v) = p(\check{v}_1 \otimes \pi(e_K)v) = p(\check{\pi}_1(\check{e}_K)\check{v}_1 \otimes v)$$

where $\check{e}_K$ is the image of e_K under the antiautomorphism $g \mapsto g^{-1}$.
But $\check{e}_K = e_K$ & so $\varphi(v)(\check{v}_1) = \varphi(v)(\check{\pi}_1(e_K)\check{v}_1)$.
Hence $\varphi(v)$ factorizes through $\check{\pi}_1(e_K)$. We have a natural injection $V_1 \otimes V_2' \to \mathrm{Hom}(\check{V}_1, V_2')$.
Since π_1 is admissible, the image of this injection is $\{f \in \mathrm{Hom}(\check{V}_1, V_2') \mid \exists$ open compact sbg $K \le G$ ∋ f factors through $\check{\pi}_1(e_K)\}$.
Thus, φ factors by $\varphi': V \to V_1 \otimes V_2'$.

I] φ' is injective.

Let $v \in V \setminus \{0\}$. Since $\bigcap_{f \in \mathrm{Hom}_{G_1}(V, V_1)} \ker f = 0$ by hypothesis, $\exists f \in \mathrm{Hom}_{G_1}(V, V_1)$ ∋ $f(v) \neq 0$.
Fix f & $\check{v}_1 \in \check{V}_1$ ∋ $\check{v}_1 \circ f(v) \neq 0$. By functoriality, f defines a function $f': (\check{V}_1 \otimes V)[G_1] \to (\check{V}_1 \otimes V_1)[G_1] \cong \mathbb{C}$. We have $f' \circ p(\check{v}_1 \otimes v) = \check{v}_1 \circ f(v) \neq 0$.
Thus, $p(\check{v}_1 \otimes v) \neq 0$ & hence $\varphi(v) \neq 0$. Thus φ is injective. As φ factors through φ', φ' is also injective proving [I]. (Pf continues next page)

By $\boxed{\mathrm{I}}$, V identifies as a $G_1 \times G_2$ sub-module of $V_1 \otimes V_2'$. By the previous lemma, $\exists$ a G_2-invariant subspace $V_2 \subseteq V_2' \ni V \cong V_1 \otimes V_2$. $\boxed{}$

The Local Theta Correspondence

Let (H_1, H_2) be a reductive dual pair of $Sp(W)$ & $(\pi_1, V_1) \in R_\psi(\tilde{H}_1)$.
Set $S(\pi_1) := \bigcap_{f \in Hom_{\tilde{H}_1}(S, V_1)} Ker f$ & $S[\pi_1] := S / S(\pi_1)$.

$S(\pi_1)$ is $\tilde{H}_1$-stable (as each $Ker f$ is) & $\tilde{H}_2$-stable ($\tilde{H}_2$ permutes f's since $C_{\widetilde{Sp}(W)}(\tilde{H}_1) = \tilde{H}_2$).
Thus, we have a quotient rep. of $\tilde{H}_1 \times \tilde{H}_2$ on $S[\pi_1]$. By the previous lemma, $\exists$ (π_2', V_2') smooth rep. of $\tilde{H}_2$, unique up to isomorphism, $\ni S[\pi_1] \cong V_1 \otimes V_2'$.

Def: We write $\pi_2' = \Theta(\pi_1)$ is the big theta lift of π_1.

Conjecture: If F is local, non-Archimedean, then $\exists!$ subspace V_2'' of V_2' which is $\tilde{H}_2$-stable $\ni$ V_2' / V_2'' is irreducible.

This conjecture has actually been proven.

Def: Let $V_2 = V_2' / V_2''$ & (π_2, V_2) be the corresponding rep. of $\tilde{H}_2$. We say π_2 is the (little) theta lift of π_1 & write $\theta(\pi_1) = \pi_2$. Alternatively, we say π_2 corresponds to π_1 & write $\pi_2 \longleftrightarrow \pi_1$. This is the theta correspondence.

Another reformulation

Let (π_1, V_1) be any irreducible admissible rep. of H_1 $\left(\begin{array}{l}\text{not necessarily satisfying} \\ \pi \circ i\partial(z) = z \cdot id_V\end{array}\right)$
If $\pi_1 \circ i\partial(z) \neq z \cdot id_V \ \forall z \in \mathbb{C}^\times$, set $\Theta(\pi_1) = 0$.

Note: $\omega_\psi(z_1, z_2) = z_1 z_2 \, Id_S \quad \forall z_1, z_2 \in \mathbb{C}^\times \left(\begin{array}{l}\hookrightarrow \widetilde{H}_1 \\ \hookrightarrow \widetilde{H}_2\end{array}\right)$. So, if π_1 doesn't satisfy the above condition, we'll have $\overline{\Theta}(\pi_1) = 0$ & hence $\Theta(\pi_1) = 0$.

We can reformulate the previous conjecture as follows:
Let (π_1, V_1) be an irreducible admissible rep. of $\widetilde{H}_1$. Then,

1) Either $\Theta(\pi_1) = 0$ or $\Theta(\pi_1)$ is an irreducible admissible rep. of $\widetilde{H}_2$
2) If $\Theta(\pi_1) \cong \Theta(\pi_1') \neq 0$, then $\pi_1 \cong \pi_1'$. (π_1' is an irred. admissible rep. of $\widetilde{H}_1$).

One can consider restricting to H_1 & H_2. This can be done with some work which we do not present here (restriction of the metaplectic extension in [MVW] or [K]).

1) Persistance & Stable Range
   i) Given a Witt tower (G, H_n), if $\Theta_n(\pi) \neq 0$, then $\Theta_m(\pi) \neq 0 \ \forall m \geq n$.
   ii) If $G = U(W)$ & $\dim W = n$, then $\forall \pi \in Irr(G) \quad \Theta_r(\pi) \neq 0 \quad \forall r \geq 2n$
2) Chains of Supercuspidals
3) Howe Duality Conjecture for type I nonquaternionic dual pairs
4) Conservation relations
5) Global Theta Correspondence