Weil Representation

$(W, \langle \cdot,\cdot \rangle)$ symplectic $\quad 1 \neq \psi : F \to \mathbb{C}^\times$

$$H(W) = W \times F = \{ (w,t) \mid (w_1,t_1)\cdot(w_2,t_2) = (w_1+w_2,\ t_1+t_2+\tfrac{1}{2}\langle w_1, w_2\rangle) \}$$

$Z(H(W)) = \{0\} \times F.$

So, ψ is a central character.

Idea: $\forall A = A^\perp \subseteq W$ we can induce $\operatorname{ind}_A^H \psi_A \ni (\operatorname{ind}_A^H \psi_A)\big|_{\{0\}\times F} = \psi.$

$A^\perp = \{ w \in W \mid \psi(\langle w, a\rangle) = 1,\ a \in A \}.$

For us, $A = X =$ Lagrangian. We'll have $A = A^\perp$. This will give the Schrödinger model.

Stone-von-Neumann Th: If ρ, π are unitary rep's of $H(W)$ with same central character, then $\rho \cong \pi$.

Idea: Assume ρ, π are smooth for simplicity. Choose $w \in W$. This gives $W \cong \hat{W} = \operatorname{Hom}(W, S^1)$ (given compact, open topology)
A closed in $W \Rightarrow A^\perp$ closed. Also, $A^{\perp\perp} = A$ & $A^\perp \cong \widehat{(W/A)}$ (Pontryagin Duality)
$(A_1 + A_2)^\perp = A_1^\perp \cap A_2^\perp$. If $A_1^\perp + A_2^\perp$ is closed, $\quad w \mapsto \psi(\langle w, \cdot\rangle)$
then $A_1^\perp + A_2^\perp = (A_1 \cap A_2)^\perp$

Now, let $A = A^\perp$. $A \hookrightarrow A \times F = A_H \subseteq H$. Let $\psi_A = 1 \times \psi$ (just need $\psi_A\big|_{\{0\}\times F} = \psi$).

Form a rep. $(\rho, S_A) = \operatorname{Ind}_{A_H}^H (\psi_A)$. In fact, for the Heisenberg grp.,
$\operatorname{Ind}_{A_H}^H(\psi_A) = c\text{-}\operatorname{Ind}_{A_H}^H(\psi_A).$

- ρ is irreducible (nontrivial)

Ex: $X+Y$ polarization of W

$$\psi_A = 1 \times \psi.$$

$$\rho(x+y, t)\, f(y') = \psi(\langle y', x\rangle + \tfrac{1}{2}\langle y, x\rangle + t)\, f(y+y')$$

"Canonical" intertwining maps $S_{A_1}^{\psi_{A_1}} \to S_{A_2}^{\psi_{A_2}}$ $\quad \psi_{A_1} \psi_{A_2}^{-1}\big|_{A_1 \cap A_2} = a \mapsto \psi(\langle a, w\rangle)$ for some $w \in W$

$$h \longmapsto \int_{A_1 \cap A_2 \backslash A_1} f((w,0)ah)\, \psi_{A_2}^{-1}(a)\, da$$

$Sp(W)$ acts on $H(W)$ & W via $g(w,t) = (gw, t)$.

$\forall g \in Sp(W)$ we have $g \longmapsto$ a rep. (ρ_ψ^g, S) $\qquad M_g \cdot \rho_\psi(h) = \rho_\psi(g(h)) M_g$

By Stone-Von-Neumann, we have intertwining operators $M_g \rho_\psi \overset{(*)}{=} \rho_\psi^g M_g$, $M: S \to S$ bounded lin. op.

By Schur's lemma, these M_g are unique up to $\mathbb{C}^\times$.

We define $\omega_\psi : g \mapsto [M_g] \in PGL(S)$. (projective Weil rep.)

Define $\widetilde{Sp}(W)$ to be $\{(g, M_g) \in Sp(W) \times GL(S) \mid (*) \text{ holds}\}$.

$$\begin{array}{ccc} \widetilde{Sp}(W) & \xrightarrow{\text{projection}} & GL(S) \\ {\scriptstyle \text{proj.}}\downarrow & & \downarrow \\ Sp(W) & \longrightarrow & PGL(S) \end{array}$$

ω_ψ is an honest rep. $\widetilde{Sp}(W) \to GL(S)$.

(Weil rep.)

$$\begin{array}{ccc} \widetilde{Sp(W_1)} \times \widetilde{Sp(W_2)} & \to & \widetilde{Sp(W_1 \oplus W_2)} \\ \downarrow & & \downarrow \\ Sp(W_1) \times Sp(W_2) & \longrightarrow & Sp(W_1 \times W_2) \end{array}$$

th: $\exists!$ 2-fold cover $\widehat{Sp(W)}$ of $Sp(W)$.

$\hat{Sp}(W) = Der(\tilde{Sp}(W))$