Witt Decomposition & Index

Witt Decomposition) Cor: W is isometric to an orthogonal sum $W \cong mH \oplus W^0$ where W^0 is anisotropic. Moreover, the integer m & the isometry class of W^0 are completely determined by the isometry class of W.

Pf: If W is anisotropic, then $W = W^0$ & we're done. Next, suppose $\exists x \in W \ni \langle x,x\rangle = 0$. As $\langle \cdot,\cdot\rangle$ is non-degenerate, $\exists y \in W \ni \langle x,y\rangle \neq 0$. The subspace of W generated by x & y is isomorphic to H. As $\langle \cdot,\cdot\rangle$ is non-degenerate on H, $W \cong H \oplus H^{\perp}$ is an orthogonal sum. Continuing inductively, we obtain $W \cong mH \oplus W^0$ where W^0 is anisotropic.

(Unique) Suppose $W \cong mH \oplus W^0 \cong nH \oplus W'$ where W^0, W' are anisotropic. Assume $m \leq n$. Then, we have an injective linear map $f: mH \to nH \subseteq W$ which takes H's to H's in the obvious manner. Also, $\langle f(x), f(y)\rangle = \langle x,y\rangle \ \forall x,y \in mH$. So, by Witt's theorem, f extends to an isometry on W. Furthermore, this isometry "fixes" mH. So, $W/_{mH} \cong W^0 \cong (n-m)H \oplus W'$. But, W^0 is anisotropic. Since H contains an isotropic vector, we must have $n = m$ & hence $W^0 \cong W'$. $\boxtimes$

Def: The integer m in the Witt decomposition $W \cong mH \oplus W^0$ is called <u>the Witt index of W.</u>

Remark: Witt's decomposition reduces the classification of ε-Hermitian spaces to the classification of anisotropic ε-Hermitian spaces. This is equivalent to determining the Witt group.

Prop: Let V be a totally isotropic subspace of W. Then V is contained in a hyperbolic subspace of W of Witt index $\dim V$.

Pf: Let $\{v_1, \dots, v_k\}$ be a basis for V. $\langle \cdot, \cdot \rangle$ is non-degenerate on W & so $\exists\, w \in W \ni$ $\langle v_1, f \rangle \neq 0$ but $\langle v_i, f \rangle = 0$ $\forall i \geq 2$. Now the subspace generated by v_1 & f is isometric to H. ~~Also, $\langle \cdot, \cdot \rangle$ is nondeg~~ So, $W \cong H \oplus H^{\perp}$ as an orthogonal sum. Furthermore, $\operatorname{span}\{v_2, \dots, v_k\}$ is a totally isotropic subspace of $H^{\perp}$ & $\dim H^{\perp} < \dim W$. Induction on $\dim W$ gives the prop. $\square$

Cor: Any maximal totally isotropic subspace of W has dimension equal to the Witt index of W.

~~The previous proof shows~~

Pf: Let V be a totally isotropic maximal subspace. By the previous proposition, $V \subseteq nH \subseteq W$. By Witt's decomposition, $W \cong mH \oplus W^0$ where W^0 is anisotropic. ~~If V is not maximal,~~ If $n < m$, we can take an isotropic vector from H to get a new totally isotropic subspace (by attaching the new isotropic vector to V). But V is maximal ↯ so $n \geq m$. Since $V \subseteq nH \subseteq mH \oplus W^0$ & W^0 is anisotropic, we must have $n = m$. The pf of the previous proposition shows $\dim V = m$. $\square$

Def: If W is hyperbolic, then a maximal totally isotropic subspace is called a <u>Lagrangian of W</u>.

§10-15 deals w/ invariants & classification of ε-Hermitian spaces. We will skip this for now.

§16 Hermitian Tensor Products

Def: Let D_1 be a division ring with the center $Z(D_1)$ containing F' (center of D). which is also a right D-module

Let W_1 be a right D_1-module & W_2 be a left D_1-module. Suppose W is an ε-Hermitian right D-module of finite dimension & let $W = W_1 \otimes_{D_1} W_2$ as a tensor product of modules. If the algebras (see §1) $B = \mathrm{End}_{D_1} W_1$ & $B' = \mathrm{End}_{(D_1, D)} W_2$ are stable under the adjoint involution of $A = \mathrm{End}_D(W)$, we say that this is a <u>Hermitian tensor product</u>.

modules hom's that respect W_2 as both a left & right D_1 & D module

So, W is an ε_1-Hermitian right D_1-module & W_2 is an ε_2-Hermitian right D_2-module, where D_2 is in the Brauer class of $D_1^0 \otimes_{F'} D$ whose center is the same as D_1's. The ε-Hermitian structures of W_1 & W_2 define the ε-Hermitian structure of W, up to similitude.

Prop: $D_1^0 \otimes_{F'} D \cong M(r, D_2)$, $A \cong M(n, D^0)$, $B \cong M(n_1, D_1^0)$, & $C = \mathrm{End}_{D_2} W_2 \cong M(n_2, D_2^0)$

$n = n_1 n_2 d_1 r^{-1}$, $n^2 d = n_1^2 d_1 \times n_2^2 d_2$, & $d_0 d_2 = r d_2$

where $n = \dim_D W$, $n_1 = \dim_{D_1} W_1$, $n_2 = \dim_{D_2} W_2$, $d_1 = \dim_{F'} D_1$, & $d_2 = \dim_{F'} D_2$.

Hermitian tensor products will be useful when we study decompositions of Symplectic spaces in §20.