The Size of Edge Chromatic Critical Graphs with Maximum Degree 7

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Abstract

In 1968, Vadim Vizing conjectured that for any edge chromatic critical graph with maximum degree Δ, m edges, and n vertices $m \geq \frac{1}{2}[(\Delta-1)n+3]$. This conjecture has attracted much attention, but it is only known to be true for $\Delta \leq 6$. We give a new lower bound for the size of 7-critical graphs.

Background

Perhaps the most famous result in Graph Theory is the Four Color Theorem, which states that every map can be colored with four colors such that no two adjacent regions have the same color. In 1880, Peter Tait introduced the concept of edge coloring in attempt to approach this problem.

Throughout this poster, G will be a simple graph with n vertices, m edges, and maximum degree Δ. An *edge coloring* of a graph is a function that assigns values to the edges of a graph such that any two adjacent edges receive different colors. G is *edge $k-colorable$* if there is an edge coloring of G with colors from $\{1, \ldots, k\}$. The edge chromatic number of G (denoted $\chi'(G)$) is the smallest k such that G is edge k-colorable. In 1965, Vizing showed that either $\chi'(G) = \Delta$ (class 1) or $\chi'(G) = \Delta + 1$ (class 2). G is said to be *critical* if it is connected, class 2, and $G - e$ is class 1 for all $e \in E(G)$. A critical graph with maximum degree Δ is called a *$\Delta - critical$ graph*. In 1968, Vizing [2] conjectured the following.

Conjecture 1 *If G is a Δ-critical graph, then $m \geq \frac{1}{2}[(\Delta-1)n+3]$, or equivalently,*

$$d_{ave} = \frac{2m}{n} \geq \Delta - 1 + \frac{3}{n},$$

where d_{ave} is the average degree of G.

Conjecture 2 *If G is a Δ-critical graph then*

$$\alpha(G) \leq \frac{n}{2}$$

where $\alpha(G)$ is the independence number of G.

Conjecture 1 is known to be true for $\Delta \leq 6$ [1], but a complete solution seems far from being found. In fact, the difference between the conjectured bounds and best lower bounds tends toward infinity as $\Delta \to \infty$.

Conjecture 1 has some interesting consequences. First, Conjecture 1 implies Conjecture 2 asymptotically. The conjecture also implies no 7-critical graph is planar. It is already known that planar graphs with maximum degree 7 are class 1, but achieving a bound of $d_{ave} \geq 6$ would provide an alternate proof. We get closer to achieving this bound by proving $d_{ave} \geq 6 - \frac{1}{7}$.

Methodology

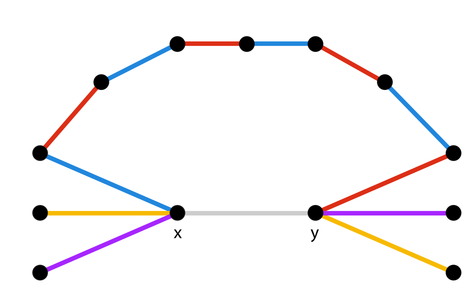
We use the discharging method. The discharging method is a technique in which each vertex, edge, and face is assigned an initial charge. The charge is then reallocated locally while conserving the total charge of the graph. The reallocated charge then confirms or contradicts some structural property.

In this problem, we suppose that a 7-critical graph G has $d_{ave} < 6 - \frac{1}{7}$ then reach contradiction. If $d_{ave} < 6 - \frac{1}{7}$, then

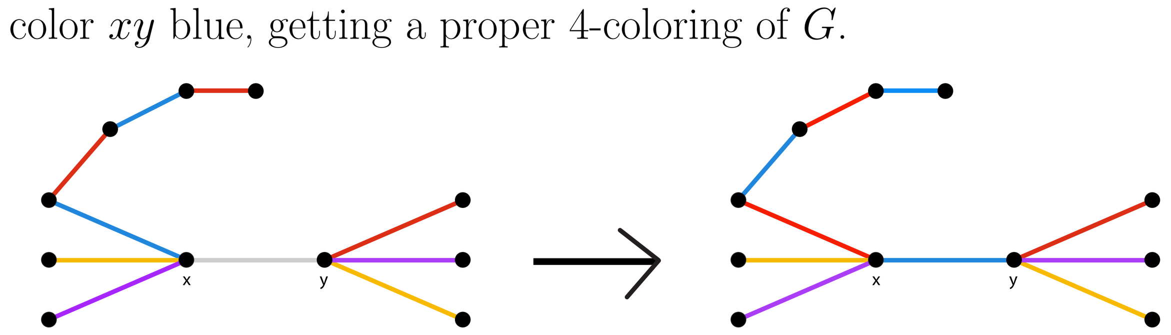
$$\sum_{x \in V} (d(x) - 6 + \frac{1}{7}) = \sum_{x \in V} M(x) < 0,$$

where $M(x) = d(x) - 6 + \frac{1}{7}$ is the initial charge of x for $x \in V$. We then reallocate the charge by a list of discharging rules that conserve the total charge. In doing so, we will get that each vertex has a nonnegative charge, contradicting our assumption.

The rules by which we discharge come from structural properties of critical graphs. A simple example follows, while some of the most important structural results detailed in the Adjacency Lemmas section below. Let G be a 4-critical graph and $xy \in E$. Then $G - xy$ is edge 4-colorable. Say x does not *see* the color red, while y does not *see* the color blue. Then the path longest path starting at x that alternates colors between blue and red must end at y.



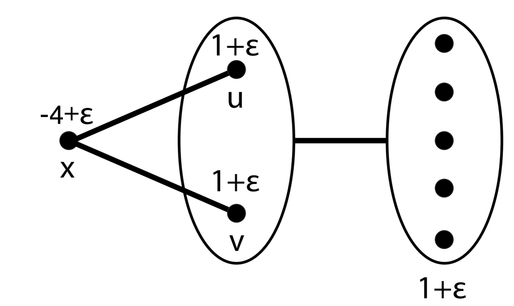
If this were not the case, we could swap colors along this path, and color xy blue, getting a proper 4-coloring of G.



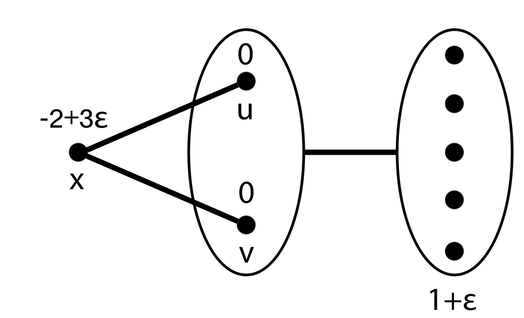
This contradicts that G is 4-critical.

An Example

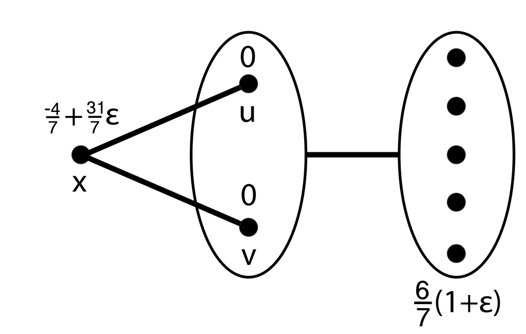
Let G be a 7-critical graph, and assign an initial charge $M(x) = d(x) - d_{ave} = d(x) - 6 + \epsilon$. Note that only degree 6 and degree 7 vertices have nonnegative charge for small ϵ. Consider a degree 2 vertex x with neighbors u and v. By Lemma 1, u and v have degree $\Delta = 7$, and by Lemma 2 (1), $N(N(x))$ all have degree 7. So we have $M(x) = -4 + \epsilon$ and $M(u) = M(v) = 1 + \epsilon$.



Our goal is to redistribute the charge such that each vertex has a nonnegative charge. So we will send the charge of u and v to x. Thus x has a new charge of $M'(x) = M(x) + M(u) + M(v) = -2 + 3\epsilon$.



Since $N(N(x))$ are also all degree 7 vertices, we can send some of that charge to x. Each u and v are adjacent to at least 5 other degree 7 vertices. Make each of these send $\frac{1}{7}$ of its charge to x. So $M''(x) = M'(x) + \frac{10}{7}(1+\epsilon) = \frac{-4}{7} + \frac{31}{7}\epsilon$.



If $M''(x) \geq 0$, then $\epsilon \geq \frac{4}{31}$. However, there is one configuration that, with out current arsenal of adjacency lemmas, requires $\epsilon \geq \frac{1}{7}$.

Adjacency Lemmas

There are two main results that were used to formulate the discharging rules. They aren't all that is necessary, but they are very useful.They are as follows.

Lemma 1 (Vizing's Adjacency Lemma) *Let x be a vertex of a Δ-critical graph. Then*

(1) if $d_k(x) \geq 1$, then $d_\Delta(x) \geq \Delta - k + 1$;

(2) $d_\Delta(x) \geq 2$

where $d_k(x)$ denotes the number of vertices of degree k adjacent to x.

Lemma 2 *Let G be critical, $xy \in E$, and $d(x)+d(y) = \Delta+2$. Then the following hold:*

(1) every vertex of $N(x,y) - \{x,y\}$ has degree Δ.

(2) every vertex of $N(N(x,y)) - \{x,y\}$ has degree at least $\Delta - 1$.

(3) if $d(x), d(y) < \Delta$ then every vertex of $N(N(x,y)) - \{x,y\}$ has degree Δ.

Beyond these two adjacency lemmas, there are other results that give more insight on the structure of critical graphs.

Results

Theorem 1 *If G is a 7-critical graph, then $d_{ave} \geq 6 - \frac{1}{7}$.*

Sketch of Proof:
Suppose not. Then $d_{ave} < 6 - \frac{1}{7}$, and

$$\sum_{x \in V} (d(x) - 6 + \frac{1}{7}) = \sum_{x \in V} M(x) < 0,$$

where $M(x) = d(x) - 6 + \frac{1}{7}$ is the initial charge of x. We then reassign a new charge, denoted $M'(x)$, to each vertex x by the following discharging rule:

(R1) Equidistribution. Each vertex x of degree 6 or 7 sends $\frac{1}{d_{\leq 5}(x)}M(x)$ to each adjacent vertex of degree at most 5 if $d_{\leq 5}(x) \neq 0$.

We then discharge $M'(x)$ according to the following rules:

(R2) Let x be a degree 2 vertex with neighbors u and v. Each degree 7 vertex $w \neq v$ adjacent to u sends $\frac{1}{7}M(w)$ to x via u and each vertex $w \neq u$ adjacent to v sends $\frac{1}{7}M(w)$ to x via v,

(R3) Let x be a degree 3 vertex adjacent to a degree 6 vertex u. Each degree 7 vertex w adjacent u but not x sends $\frac{1}{7}M(w)$ to x via u,

(R4) Let x be a degree 4 vertex with a degree 5 neighbor y. Then y sends ϵ to x,

(R5) Let x be a degree 4 vertex adjacent to a degree 6 vertex y. Each degree 5 vertex distance two away from x send $\frac{1}{8}\epsilon$ to x.

By carefully exhausting possible configurations within G (which is limited by several adjacency lemmas and known structural properties of critical graphs), we can conclude each vertex now has a nonnegative charge. However, since the charge was conserved, we arrive at a contradiction.

Further Research

The bound of $d_{ave} \geq 6 - \frac{1}{7}$ is reliant on one configuration. Future work could include developing an adjacency lemma that will loosen this requirement and produce a better bound.

References

References

[1] R. Luo, L. Miao, and Y. Zhao. The size of edge chromatic critical graphs with maximum degree 6. *Journal of Graph Theory*, 60(2):149–171, 2009.

[2] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz No.*, 3:25–30, 1964.