MA 460 Supplement: Ceva's theorem using analytic geometry

Donu Arapura

Ceva's theorem, which is theorem 36 in McClure, says

Theorem 36. *Suppose that $\triangle ABC$ is a triangle, and let A', B' and C' be points on the lines BC, AC and AB other than the vertices. If AA', BB', CC' are concurrent, then*

$$\frac{\vec{A'B}}{\vec{A'C}}\frac{\vec{B'C}}{\vec{B'A}}\frac{\vec{C'A}}{\vec{C'B}} = -1 \tag{1}$$

Recall that the signed ratios occurring above are plus or minus the usual ratios (see section 7.4 of McClure.) The above equation is equivalent to the equation

$$\frac{A'B}{A'C}\frac{B'C}{B'A}\frac{C'A}{C'B} = 1 \tag{2}$$

and the condition that either that all three points A', B', C' lies inside the triangle (case 1) or exactly one of A', B', C' lies inside (case 2). We will concentrate on the first case in the following discussion.

We have also a converse.

Theorem 37. *If A', B', C' are points chosen as above such that equation* (1) *holds, then AA', BB', CC' are concurrent.*

We start by proving the last theorem in the case 1 that A', B', C' lie inside the triangle.

Proof. As we did in class, we can write

$$A' = a_1B + a_2C$$

$$B' = b_1C + b_2A$$

$$C' = c_1A + c_2B$$

where the coefficients are positive with $a_1+a_2 = b_1+b_2 = c_1+c_2 = 1$. Also as we worked out in class, $A'B = d(A', B) = ||(a_1 - 1)B + a_2C|| = a_2||B - C||$ etc. After substituting into equation (2) we get

$$\frac{a_2}{a_1}\frac{b_2}{b_1}\frac{c_2}{c_1} = 1 \tag{3}$$

We now want to guess an expression for the intersection of all three lines AA', BB', CC'. Set

$$X = \alpha A + \beta B + \gamma C$$

Choosing $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$ guarantees that X lies in the triangle $\triangle ABC$. In order to guarantee that X lies on the line segment AA', we need to be able write

$$X = tA + sA'$$

with positive coefficients satisfying $t + s = 1$. Equating

$$\alpha A + \beta B + \gamma C = X = tA + sa_1B + sa_2C$$

shows that we need the ratio $\beta/\gamma = a_1/a_2$. We have similar conditions for X to lie on BB' and CC'. It's not apriori obvious that these conditions are consistent, i.e. that we won't arrive at a contradiction. But we don't because of (3). Let us summarize things.

Lemma 1. *We can choose positive numbers α, β, γ such that $\alpha + \beta + \gamma = 1$ and*

$$\frac{\beta}{\gamma} = \frac{a_1}{a_2} \tag{4}$$

$$\frac{\gamma}{\alpha} = \frac{b_1}{b_2} \tag{5}$$

$$\frac{\alpha}{\beta} = \frac{c_1}{c_2} \tag{6}$$

Proof of lemma. Setting $\beta = sa_1$ and $\gamma = sa_2$ forces the first equation (4) to hold. Let $\alpha = t$. Then $t + s = \alpha + \beta + \gamma = 1$. Therefore $\gamma = (1 - t)a_2$. Substituting into (5) above yields

$$\frac{(1-t)a_2}{t} = \frac{\gamma}{\alpha} = \frac{b_1}{b_2}$$

so that

$$t = \frac{a_2}{\frac{b_1}{b_2} + 1}$$

Therefore

$$\alpha = t, \beta = (1 - t)a_1, \gamma = (1 - t)a_2,$$

with t as above, satisfies the (4) and (5). The only thing remaining to check is the last equation (6). From what we already know, we see that

$$\frac{\alpha}{\beta} = \frac{\alpha}{\gamma}\frac{\gamma}{\beta} = \frac{b_2}{b_1}\frac{a_2}{a_1}$$

By (3), this equals $\frac{c_1}{c_2}$. It is worth observing that in fact (6) holds if and only if (3) holds. □

The lemma implies that $X = \alpha A + \beta B + \gamma C$ lies on the intersection of AA', BB' and CC'. Therefore these lines are concurrent.

□

To prove case 1 of Ceva's theorem, we can use a similar set up but work backwards in some sense. We choose X so that (4) and (5) hold. We can then argue, as above, that we can choose a possibly different point C'' on AB for which (6) and therefore (3) hold. Then we would have to show $C' = C''$. We are being a bit vague about the details since this will be a homework problem.