MA 460 Supplement: spherical geometry

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Although spherical geometry is not as old or as well known as Euclidean geometry, it is quite old and quite beautiful. The original motivation probably came from astronomy and navigation, where stars in the night sky were regarded as points on a sphere. To get started, let S be the sphere of radius 1 centered at the origin O in three dimensional space. Using xyz coordinates, we can place O at $(0, 0, 0)$, which means $x = y = z = 0$, then S is given by $x^2 + y^2 + z^2 = 1$. Points will now be understood as points on S. A line is now a great circle. This is a circle obtained by intersecting S with a plane through O. (When we need to talk about lines in the usual sense, we will call them ordinary lines or straight lines. We also occasionally refer to a great circle as a spherical line when we want to be absolutely clear.) Given points P and Q, the spherical distance between them is the angle $\angle POQ$ measured in radians which we use from now on for all angles (recall π radians $= 180°$). Given points A, B, C on S, we define the spherical angle $\angle ABC$ as follows. Let ℓ be the ordinary line in space tangent to the spherical line (i.e. circle) AB at B, similarly let m be tangent to CB at B. Then $\angle ABC$ is the angle between ℓ and m. A (spherical) triangle is given by points A, B, C and (spherical) line segments AB, AC, BC connecting them. We can define the area of $\triangle ABC$ the way we would in a calculus class as an integral, although we will generally use more elementary methods. The one fact we will need from calculus, however, is that $area(S) = 4\pi$.

We won't use an axiomatic approach, but instead we combine analytic and other methods. However, once we establish a collection of basic facts, we can sometimes argue as we did in Euclidean geometry.

1 Elementary properties

Let us compare some of the basic facts from McClure with what is happening in spherical geometry. BF6 "the whole is the sum of the parts" still holds here and for any other kind of geometry. BF10 on the existence of midpoints is true and we prove it next. (To avoid conflicting with earlier theorem numbers, we start with theorem 100.)

Theorem 100. *If AB is a spherical line segment from A to B there is a point M on AB, such that the spherical distances between A and M, and B and M, are equal.*

Proof. AB is given by an arc on great circle on a plane P containing A, B, and O. Let M be the point on this arc meeting the ray on P which bisects the angle $\angle AOB$. $\square$

BF12 also works (with a curious twist):

Theorem 101. *Given a spherical line ℓ and a point A not on the line, there exists a line m which meets ℓ at right angles ($\pi/2$ radians) in two points.*

Proof. Let ℓ be given by intersecting a plane L with S. Choose a plane M through A which is perpendicular to L, and let B be the point where it meets L. Let m be the intersection of M with S. This will be perpendicular to ℓ at both points of intersection. $\square$

BF7 "through two points there is one and only one line" is partly true but not completely. It is true that through any pairs of points there is a spherical line, but there may be more than one. Let us explain. Given A, the antipode A' is the point on the opposite side of S where the straight line AO meets it. In coordinates if $A = (x, y, z)$, $A' = (-x, -y, -z)$. Any plane through the ordinary line AA' passes through O, and therefore cuts out a great circle. Therefore, there are infinitely many lines through A and A'.

Theorem 102. *Given points A and B there exists a spherical line containing them. If A and B are antipodes, there are infinitely many lines containing them. If A and B are not antipodes, then the line is unique.*

Proof. A spherical line containing A and B exists because by intersection S with the plane L passing through O A and B. If A and B lie on two spherical lines, then O, A, B lie on two planes L and M. Then these points will lie on the straight line given by the intersection of L and M. This means that A and B are antipodes. Conversely, if A and B are antipodes, then all three points lie on a straight line ℓ. There are infinitely many planes containing ℓ. $\square$

Finally, let us take a look at BF13 "Given a line ℓ and a point P not on ℓ, it is possible to draw line through P parallel to ℓ". This turns out to be completely false.

Theorem 103. *Any two spherical lines meet.*

Therefore there are *no* parallel lines at all. Thus spherical geometry is really quite different, and these differences are interesting. Nevertheless, we will see that many things do work as before.

2 Spherical triangles

We now want to summarize some basic facts about spherical triangles, that we can use in homework. First, we need to be bit more precise on what we mean by a triangle. Given three points A, B, C, and spherical lines connecting them, it divides the sphere S into two regions. We require that one of these regions has all internal angles strictly less than π. We regard these angles as the angles of the triangle, and this region as the inside of the triangle. As we will see we have big difference with Euclidean geometry: *the sum of angles of a spherical triangle is never π radians (*$180°$*).* On the plus side it will turn out that many basic facts do still hold. First we need to give the definition. Two spherical triangles $\triangle ABC$ and $\triangle DEF$ are *congruent* if the corresponding lengths and angles are equal. To be more explicit $AB = DE, AC = DF$ etc. Then we show that SSS, SAS, ASA, stated exactly as before, still hold. Surprisingly, there is completely new fact AAA, which says that if two triangles are congruent if their corresponding angles are equal. This is false in Euclidean geometry because we can have similar triangles which are not congruent. Let us see how to use these facts.

Problem Prove that a spherical triangle has two equal sides if and only if it has two equal angles.

This is the analogue of theorem 5 of McClure. We can carry out one half of the proof almost word for word. Let $\triangle ABC$ be a triangle such that $AC = BC$. Let M be the midpoint of BC which exists by theorem 100. Choose a line AM using theorem 102. Using SSS, we can conclude that $\triangle AMC$ and $\triangle BMC$ are congruent. Therefore $\angle A = \angle B$.

Conversely, suppose that $\angle A = \angle B$, we have to show that the opposite sides are equal. The old proof, which relied on the fact that the sum of the angles of a triangle is $180°$, no longer works. We use a completely new strategy. The argument is short but possibly confusing in that it uses a new tool AAA, and it uses it in a tricky way. Comparing $\triangle ABC$ and $\triangle BAC$, we see that $\angle A = \angle B$, $\angle B = \angle A$ and of course $\angle C$ is the same. Thus $\triangle ABC$ and $\triangle BAC$ are congruent. Note that this congruence interchanges A and B. The conclusion is that $AC = BC$.

3 Sum of angles of a triangle

Theorem 3 of McClure that the sum of angles of a triangle is π radians is false. The correct replacement for it is somewhat surprising.

Theorem 104 (Gauss-Bonnet)**.** *If $\triangle ABC$ is a spherical triangle,*

$$\angle A + \angle B + \angle C = \pi + area(\triangle ABC)$$

Corollary 1. *Two congruent spherical triangles have the same area.*

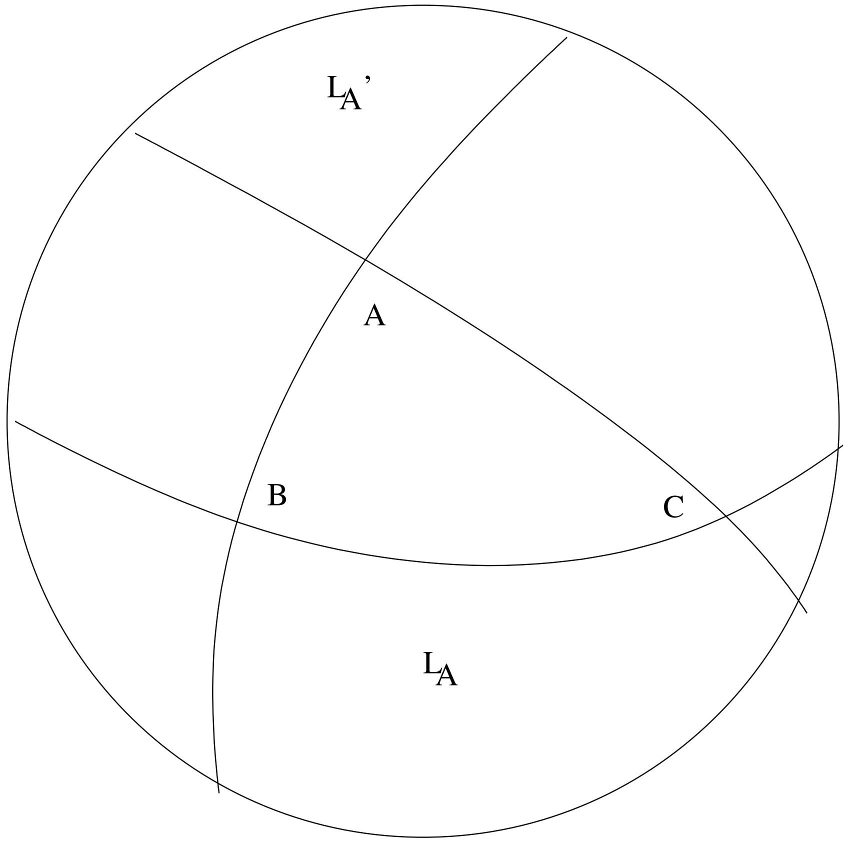
Gauss and Bonnet proved a much more general theorem valid for any curved surface. The special case above has an elementary proof that we will explain. We start by explaining what a lune is. It is basically a spherical polygon with 2 sides. Of course such a thing cannot exist in the Euclidean plane, but it can on S. Given two antipodal points A and A', choose any pair of great circles through A and A'. This divides S into 4 regions each of which looks like an orange slice. These are called lunes. Given a lune, choose a point B and C on each side, the angle $\angle BAC$ is the angle of the lune.

Theorem 105. *The area of a lune L with angle α is 2α.*

Proof. This is easy enough using formulas from calculus, but we prefer to give a more self-contained proof. Suppose that $\alpha = 2\pi/q$, where q is an integer. Then we subdivide the sphere into q lunes each with angle $2\pi/q$. Therefore the area of a single lune is $1/q$ times the area of S, which is $4\pi/q = 2\alpha$. Suppose that $\alpha = 2\pi(p/q)$, where p and q are integers, then again of L_α the area is p times the area of $L_{2\pi/q}$. So once again the formula holds. In general, we can write $\alpha = 2\pi \lim r_n$, where r_n is a sequence of rational numbers. Then $areaL_\alpha = \lim_{n\to\infty} areaL_{2\pi r_n} = \lim 2\pi r_n = 2\pi\alpha$. $\square$

Proof of theorem 104. Let A', B', C' be the antipodes of A, B, C respectively. They form a triangle with the same area. Let L_A be the lune bounded by the lines AB and AC and containing B and C. Similarly, let L_B and L_C be the lunes containing $\triangle ABC$ bounded by BA, BC and CA, CB respectively. Let L'_A, L'_B, L'_C be the antipodal lunes. These contain $\triangle A'B'C'$. If we remove $\triangle ABC$ from L_B and L_C, and $\triangle A'B'C'$ from L'_B and L'_C, we get non overlapping regions which cover S i.e.

$$S = L_A \cup (L_B - \triangle ABC) \cup (L_C - \triangle ABC) \cup L'_A \cup (L'_B - \triangle A'B'C') \cup (L'_C - \triangle A'B'C')$$



Therefore

$$area(S) = area(L_A) + [area(L_B) - area(\triangle ABC)] + [area(L_C) - area(\triangle ABC)]$$

$$+area(L'_A) + [area(L'_B) - area(\triangle A'B'C')] + [area(L'_C) - area(\triangle A'B'C')]$$

which implies

$$4(\angle A + \angle B + \angle C) - 4area(\triangle ABC) = 4\pi$$

Whence the theorem. □

4 Polar triangles

Our goal is to prove spherical ASA and AAA assuming SAS and SSS. We will finish the proof of the last two statements in the next section. The proof is based on a nice geometric construction. Given a spherical line ℓ obtained by intersection S with a plane L, let m be the straight line through O perpendicular to L. m will intersection S in two points called the poles of ℓ For example, the poles of the equator $z = 0$ are the north and south poles $(0, 0, \pm 1)$. We have

Theorem 106. *Suppose that ℓ is a spherical line and P is a point not on ℓ.*

(a) *If P a pole of ℓ, then for any point Q on ℓ, the spherical distance $PQ = \pi/2$.*

(b) *Suppose that for two points Q_1, Q_2 on ℓ we have $PQ_1 = \pi/2$ and $PQ_2 = \pi/2$ then P is a pole of ℓ.*

(c) *Suppose that P is a pole of ℓ and Q_1, Q_2 are on ℓ, then $Q_1Q_2 = \angle Q_1PQ_2$.*

Proof. If P is a pole then, essentially by definition, $PQ = \angle POQ = \pi/2$.

Now suppose that we have two points Q_1, Q_2 as in the last statement. The conditions $PQ_i = \pi/2$ for $i = 1, 2$ tell us that OP is perpendicular to the plane L containing O, Q_1 and Q_2. Since ℓ is the intersection of L with S, P must be a pole.

This proves (a) and (b). Item (c) will be a homework problem. □

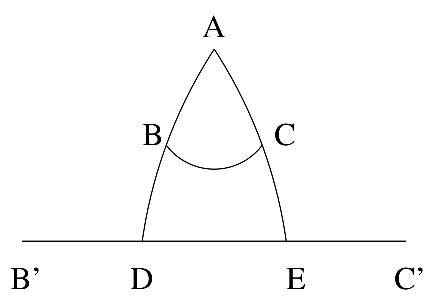
Given a spherical triangle $\triangle ABC$, the polar triangle $\triangle A'B'C'$ is the triangle with A a pole of $B'C'$ on the same side as A', B a pole of $A'C'$ on the same side as B', and C a pole of $A'B'$ on the same side as C'.

Theorem 107. *If $\triangle A'B'C'$ is the polar triangle to $\triangle ABC$, then $\triangle ABC$ is the polar triangle to $\triangle A'B'C'$.*

Proof. By assumption, B is the pole of $A'C'$. Therefore $A'B = \pi/2$ by part (a) of the previous theorem. Similarly, $A'C = \pi/2$. Applying the part (b) of last theorem, we find that A' is the pole of BC on the same side as A. Similarly B' and C' are poles of AB and AB on the appropriate sides. □

Theorem 108. *If $\triangle A'B'C'$ is the polar triangle to $\triangle ABC$, then $\angle A + B'C' = \pi$.*

Proof. Extend the lines AB and AC so that they meet $B'C'$ in points D and E.



Then $B'E = C'D = \pi/2$ by theorem 106 (a). Therefore $B'E + C'D = \pi$. We can write $B'E + C'D = B'C' + DE$. Theorem 106 (c) implies $DE = \angle A$. Combining these equations gives

$$B'C' + \angle A = \pi$$

□

The next result assumes spherical SSS which we will prove later.

Theorem 109 (AAA)**.** *If two spherical triangles have equal angles, then they are congruent.*

Proof. Let $\triangle ABC$ and $\triangle DEF$ have $\angle A = \angle D$ etc. Let $\triangle A'B'C'$ and $\triangle D'E'F$ be the polar triangle. By theorem 108,

$$B'C' = \pi - \angle A = \pi - \angle D = E'F'$$

etc. So by SSS $\triangle A'B'C'$ is congruent to $\triangle D'E'F'$. Therefore they have the same angles. Now theorem 107 implies that $\triangle ABC$ and $\triangle DEF$ are the polar triangles of $\triangle A'B'C'$ and $\triangle D'E'F'$. Thus we can apply theorem 108 with roles reversed to get

$$BC = \pi - \angle A' = \pi - \angle D' = EF$$

etc. So the original triangles are congruent. □

The following will be left as a homework problem.

Theorem 110 (Spherical ASA)**.** *If two spherical triangles have two equal angles and the included sides are also equal, then the two triangles are congruent.*

5 Spherical trigonometry

The goal in this section will be to prove the spherical versions of SSS and SAS using analytic geometry and trigonometry. So it will have a different flavour from what we have been doing up to now. The usual version of SSS can be deduced from the law of cosines:

Theorem 111. *If $\triangle ABC$ is a Euclidean triangle with sides a, b, c opposite A, B, C. Then*

$$c^2 = a^2 + b^2 - 2ab \cos \angle C$$

This tells us that

$$\angle C = \cos^{-1} \frac{c^2 - a^2 - b^2}{2ab}$$

with similar formulas for the other two angles. Thus the sides determine the angles. The trick is to establish a spherical version of this. We start with a right triangle and generalize Pythagoras' theorem and formulas for computing sines and cosines of angles. The new statement may look unrecognizable, but keep in mind that everything including lengths are now angles

Theorem 112 (Spherical Pythagoras)**.** *Let $\triangle ABC$ be a spherical triangle with $\angle C = \pi/2$. Let a, b, c be the lengths of sides opposite to A, B, C. Then*

$$\cos c = \cos a \cos b \tag{1}$$

$$\cos \angle A = \frac{\cos a \sin b}{\sin c} \tag{2}$$

$$\sin \angle A = \frac{\sin a}{\sin c} \tag{3}$$

Proof. After doing a rotation, we can assume that $C = (0, 0, 1)$ and that A lies on the xz-plane. Since BC is perpendicular to AC, B must lie on the xy-plane. Therefore, $A = (\alpha_1, 0, \alpha_3)$ and $B = (0, \beta_2, \beta_3)$ for some values $\alpha_1, \ldots$. Since the spherical distance from A to C is b, we must have $A = (\pm \sin b, 0, \cos b)$. For similar reasons, $B = (0, \pm \sin a, \cos a)$. Since A and B are unit vectors, cosine of angle between A and B, which is exactly $\cos c$, is given by the dot product

$$\cos c = 0 + 0 + \cos b \cos a$$

The proof of the second equation is a bit confusing and may be skipped. First, we need to interpret $\angle A$ in a convenient way. This is the angle between the spherical lines AB and AC. This is the same thing as the angle between the plane spanned by A and B and the xz-plane. This is also equal to the angle between normal vectors to these planes, provided that these lie in the same half space (removing the xz divides $\mathbb{R}^3$ into half spaces $y > 0$ and $y < 0$). By definition the vector cross product $A \times B$ is normal to the first plane. Recall that the length of this vector is the product of lengths of A and B with the sine of the angle between them which is $\sin c$. This is oriented by the right hand rule. The usual formula that we learn in for example Math 261, involving determinants, allows us to compute this vector as

$$A \times B = (-\cos b \sin a, -\sin b \cos a, \sin b \sin a)$$

For the normal to the xz-plane, we use $N = (0, -1, 0)$. Taking the dot product $(A \times B) \cdot N$ yields $\sin b \cos a$. On the other hand, the product is the lengths of $(A \times B)$ and N and $\cos \angle A$. Setting these equal gives

$$\sin b \cos a = \sin c \cos \angle A$$

For the third equation, we use the trigonometric identity $sin^2 \theta + \cos^2 \theta = 1$. This together with (2) implies

$$\sin^2 \angle A = \frac{\sin^2 c - \cos^2 a \sin^2 b}{\sin^2 c} = \frac{1 - \cos^2 c - \cos^2 a \sin^2 b}{\sin^2 c}$$

Now substitute (1) to get

$$\sin^2 \angle A = \frac{1 - \cos^2 a \cos^2 b - \cos^2 a \sin^2 b}{\sin^2 c} = \frac{1 - \cos^2 a}{\sin^2 c} = \frac{\sin^2 a}{\sin^2 c}$$

$\square$

Theorem 113 (Spherical law of cosines). *Let $\triangle ABC$ be a spherical triangle with sides a, b, c opposite A, B, C. Then*

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Proof. We drop a perpendicular from A to the line CB, which is justified by theorem 101. Let D be the point of intersection. Let x and h be the spherical distances from D to C and A respectively. Then $a - x$ is the distance from D to B. Applying theorem 112 to $\triangle ADB$ shows that

$$\cos c = \cos h \cos(a - x) = \cos h \cos a \cos x + \cos h \sin a \sin x \tag{4}$$

The last equality follows from trigonometry. Now applying theorem 112 to $\triangle ACD$ and rearranging terms gives

$$\cos h \sin x = \cos C \sin b \tag{5}$$

$$\cos x \cos h = \cos b \tag{6}$$

Substituting the last two equations into (4) proves that

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

□

Theorem 114 (Spherical SSS). *If two spherical triangles have equal corresponding sides, then they are congruent.*

Proof. Let $\triangle ABC$ be a triangle. Theorem 113 allows us to solve for the cosines of $\angle A, \angle B, \angle C$ using just the lengths of the sides. Since these angles lie strictly between 0 and π, the cosines determine the angles. Therefore we can recover $\angle A, \angle B, \angle C$ just in terms of the lengths of the sides. In particular, any two triangles with equal sides would also have equal angles, and would be therefore be congruent. □

Theorem 115 (Spherical SAS). *If two spherical triangles have two equal sides and the included angles are also equal, then the two triangles are congruent.*

Proof. By theorem 113 the third sides opposite the included angles are also equal. Now we can apply theorem 114 to conclude that the triangles are congruent. □