1 Complex Manifolds

Def A function $f: U \to \mathbb{C}$, $U \subseteq \mathbb{C}^n$ open, is **holomorphic** if it can be expanded as convergent power series in a nbhd of every $p \in U$.

Def An n dim'l **complex manifold** is a connected ringed space $(X, \mathcal{O}_X^{an})$, where X is metrizable, and which is locally isomorphic to $(B, \mathcal{O}_B^{an})$ where $B \subset \mathbb{C}^n$ is the open unit ball.

Rmk 1) n is the complex dim. The real dim is $2n$.

2) A Riemann surface is the same thing as a complex manifold of dim 1.

The following is a generalization of a fact from complex analysis in one variable.

Theorem (Maximum Principle)

If $f: \bar{B} \to \mathbb{C}$ is holomorphic (i.e. f is the restriction of a holomorphic function on a bigger open set), and if $|f|$ attains a maximum on B (the interior) then f is constant.

Holomorphic functions are much more rigid than C^∞. Here is an example

Theorem A holomorphic function on a compact connected* complex manifold is constant

pf: Let $f \in \mathcal{O}^{an}(X)$. Then $|f|$ attains a maximum at some $p \in X$ by compactness. Let $S = \{x \in X \mid f(x) = f(p)\}$. This is closed. It suffices to prove that S is open.

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* I sometimes forget to say this. So insert when necessary

Let $x \in S$. Let B be a small ball centered at x. Then $|f|$ takes a maximum inside $B \Rightarrow f|_B$ constant $\Rightarrow B \subset S$. Therefore S is open //

2 Regular functions on $\mathbb{P}^n_k$

Theorem A regular function on $\mathbb{P}^n_k$ is constant.

Rmk This should be viewed as an analogue of the previous theorem. Since $\mathbb{P}^n_{\mathbb{C}}$ is a compact complex manifold (with Euclidean topology).

pf We start with $n=1$. Let x_0, x_1 be homog. coordinates.

We have an open cover of $\mathbb{P}^1_k$ by $U_0 = \{x_0 \neq 0\}$, $U_1 = \{x_1 \neq 0$

We have isomorphism

$$\mathcal{U}_0 \xrightarrow{\sim} \mathbb{A}^1$$

$$[x_0, x_1] \longmapsto \frac{x_1}{x_0} = y$$

$$\mathcal{U}_1 \xrightarrow{\sim} \mathbb{A}^1$$

$$[x_0, x_1] \longmapsto \frac{x_0}{x_1} = y^{-1}$$

It follows that

$$\mathcal{O}(\mathbb{P}^1_k) = \underbrace{k[y]}_{\text{reg function on } \mathcal{U}_0} \cap \underbrace{k[y^{-1}]}_{\text{reg fun on } \mathcal{U}_1}$$

$$= k$$

For arbitrary n, we use the fact that any 2 pts of $\mathbb{P}^n_k$ can be connected by a line ($\cong \mathbb{P}^1$) to conclude the proof //

3 Projective Varieties

Given a closed $X \subset \mathbb{P}^n_k$ for the Zariski topology, the image

$$X \cap U_i \xrightarrow{\sim} X_i \subset \mathbb{A}^n_k$$

is closed. Define $f : U \to k$ to be regular if $f|_{U \cap X_i}$ is regular in the previous sense.

Theorem We have a sheaf of regular functions $\mathcal{O}_X$, and $(X, \mathcal{O}_X)$ is an algebraic variety. Furthermore if X is irreducible, then $f \in \mathcal{O}_X(X)$ is constant.

The first two statements are straightforward. We prove the last statement assuming the following basic fact:

Theorem A projective variety is complete.

We should recall.

Def A variety X is complete if the image $f(X) \subset Y$ of any morphism $f: X \to Y$, is closed.

(This should be viewed as the analogue of compactness.)

Proof of constancy of global regular functions

Let X be an irred* projective variety. $f \in \mathcal{O}_X(X)$ can be viewed as a morphism

$$f : X \to \mathbb{A}^1$$

Then $f(X)$ is closed and irred,

$\Rightarrow$ Either $f(X)$ is a pt or $f(X) = \mathbb{A}^1$

* usually included in def.

Now consider the composite
$$\bar{f} : X \longrightarrow \mathbb{A}^1 \subset \mathbb{P}^1$$
Now $\bar{f}(X)$ is closed. But $\bar{f}(x) = f(x)$

Therefore $f(X) \neq \mathbb{A}^1$, so it must be a pt. //

4 Ideal Sheaves

Let $(X, \mathcal{O}_X)$ be an algebraic variety, and let $Y \subseteq X$ be a closed set (viewed as a reduced subscheme *)

The ideal sheaf $\mathcal{I}_Y$ is
$$\mathcal{I}(U) = \{ f \in \mathcal{O}_X(U) \mid f|_{Y \cap U} \equiv 0 \}$$
identically zero

As the name suggests,
$$\mathcal{I}_Y(U) \subset \mathcal{O}_X(U)$$
is an ideal, and $\mathcal{I}_Y \subset \mathcal{O}_X$ is a subsheaf.

* You can ignore this.

Lemma We an exact sequence of sheaves

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \text{"}\mathcal{O}_Y\text{"} \to 0$$

Where

$$\text{"}\mathcal{O}_Y\text{"}(U) = \mathcal{O}_X(U \cap Y)$$

is extension by 0. (In the future we will drop the quotes.)

pf When U is affine, we have an exact sequence

$$0 \to \mathcal{I}_Y(U) \to \mathcal{O}_X(U) \to \mathcal{O}_Y(U) \to 0$$

because $\mathcal{O}_Y(U) =$ coord ring of $Y \cap U$

$$\cong \mathcal{O}_X(U) / \mathcal{I}(Y \cap U)$$

by what we said the first week

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Cor We have an exact sequence

$$0 \to \mathcal{I}_Y(X) \to \mathcal{O}_X(X) \to \mathcal{O}_Y(X)$$

Ex The last map need not be surjective:

Let $X = \mathbb{P}^1_k$ and $Y = \{p, q\}$, $p \neq q$.

Then

$$\begin{array}{ccc} \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_Y(X) \\ \| & & \| \\ k & & k \oplus k \end{array}$$

which isn't surjective.

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5 Intro to sheaf cohomology

Given a presheaf $\mathcal{F}$ on X set

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$

Our goal is to eventually prove:

Theorem There exists a sequence of functors
$$H^n(X,-): PAb(X) \to Ab, \quad n=0,1,\ldots$$
such that for any exact sequence of sheaves
$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$
we get a long exact sequence
$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{A}) \arrow[r] & H^0(X,\mathcal{B}) \arrow[r] & H^0(X,\mathcal{C}) \arrow[dll, "\delta"] \\ & H^1(X,\mathcal{A}) \arrow[r] & \cdots \end{tikzcd}$$

The maps labelled δ, called connecting maps, are canonical.

We will see further properties as we proceed.

There are a few approaches

(1) Čech cohomology: we'll get to this later. It is concrete and fairly computable but it is has certain limitations. [Many books use this.]

(2) Derived functors: These were invented by Cartan and Eilenberg in the mid 1950's, and applied to define sheaf cohology (correctly) by Grothendieck in 1957 [Hartshorne, chap III, describes this.]

(3) Canonical flasque resolution: This approach is due to Godement 1958. I'll use a simplified version of this.

6 Flasque Sheaves

Def A sheaf $\mathcal{F}$ is **flasque** if $\forall\, U \in \mathrm{Open}(X)$, $\rho_{XU} : \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. ["Flasque" is a French word sometimes translated as "flabby".]

Ex Given an abelian group A and $p \in X$, the skyscraper sheaf

$$A_p(U) = \begin{cases} A & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}$$

with obvious restriction is flasque.

Theorem If

$$0 \to \mathcal{A} \to \mathcal{B} \xrightarrow{\eta} \mathcal{C} \to 0$$

is an exact seq. of sheaves and $\mathcal{A}$ is flasque, then

$$\eta_X : \mathcal{B}(X) \to \mathcal{C}(X)$$

is surjective.

<u>pf</u> Let $\gamma \in \mathcal{C}(X)$. By Zorn's lemma, we can find a maximal open set $U \subseteq X$ such that $\gamma|_U$ lies in the image of $\mathcal{B}(U)$. Let $\beta \in \mathcal{B}(U)$ map to $\gamma|_U$.

Suppose $U \neq X$. Choose $p \in X \setminus U$. Then γ_p lies in the image of $\mathcal{B}_p$. Therefore $\exists$ an open nbhd V of p and $\sigma \in \mathcal{B}(V)$ s.t. $\eta(\sigma) = \gamma|_V$

The difference

$$\alpha' = \beta|_{U \cap V} - \sigma|_{U \cap V} \in \mathcal{A}(U \cap V)$$

Since $\mathcal{A}$ is flasque, we can lift α' to $\alpha \in \mathcal{A}(V)$

The sections β on U and $\sigma + \alpha$ on V now patch to a section $\tilde{\beta} \in \mathcal{B}(U \cup V)$ which maps to γ.

This contradicts maximality so $X = U$. //

Given a sheaf $\mathcal{F}$, let

$$G(\mathcal{F}) = \bigoplus_{p \in X} \mathcal{F}_p ,$$

where the stalks are viewed as skyscraper sheaves.
Note that $G(\mathcal{F})$ is flasque

We a morphism

$$\mathcal{F} \longrightarrow G(\mathcal{F})$$

by $\quad f \longmapsto (f_p)_p$

Since $f_p = 0 \ \forall p \Rightarrow f = 0$

This is an injective morphism or more accurately a monomorphi

Define
$$C(\mathcal{F}) = \left(G(\mathcal{F}) / \mathcal{F} \right)^{+}$$
Then we have an exact sequence of sheaves:
$$0 \to \mathcal{F} \to G(\mathcal{F}) \to C(\mathcal{F}) \to 0$$

Def

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$$
$$H^1(X, \mathcal{F}) = \operatorname{coker}\left(\Gamma(G(\mathcal{F})) \to \Gamma(C(\mathcal{F}))\right)$$
$$H^{n+1}(X, \mathcal{F}) = H^1(X, C^n(\mathcal{F}))$$
where $C^n(\mathcal{F}) = \underbrace{C(C(\cdots C}_{n \text{ times}}(\mathcal{F})\cdots))$