Chapter 1

Riemann surfaces

1.1 Hyperelliptic integrals and curves

As all of us learn in calculus, that integrals involving square roots of quadratic polynomials can be evaluated by elementary methods. For higher degree polynomials, this is no longer true, and this was a subject of intense study in the 18th and 19th centuries by Euler, Legendre, Abel,.... An integral of the form

$$\int \frac{p(x)}{\sqrt{f(x)}} dx \tag{1.1}$$

is called elliptic if $p(x)$ is polynomial and $f(x)$ is a polynomial of degree 3 or 4, and hyperelliptic if f has higher degree.

A big advance in the above study involved switching from real to complex analysis. But the really big step was due to Riemann, who introduced the geometric point of view in the mid 19th century. He suggested that we should really be looking at the curve X^o defined by

$$y^2 = f(x)$$

in $\mathbb{C}^2$. When $f(x) = \prod(x - a_i)$ has distinct roots (which we assume from now on), X^o is a nonsingular affine algebraic curve. A bit more precisely, let

$$F(x, y) = y^2 - f(x)$$

Nonsingularity means that the gradient $\nabla f = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$ does not vanish on the zero set

$$X^o = V(F) := \{(x, y) \mid F(x, y) = 0\}$$

Since we are working over $\mathbb{C}$, we can also regard it as a Riemann surface. We will give the precise definition shortly, but intuitively it is is something which locally looks like $\mathbb{C}$. To see that it is the case for X^o, we need to invoke the implicit function theorem as explained later.

It is convenient to add points at infinity to make it a compact Riemann surface X called a (hyper)elliptic curve. To make this rigorous, we first take the projective closure of X^o

$$\overline{X} = \{[x, y, z] \in \mathbb{P}^2_{\mathbb{C}} \mid F^h(x, y, z) = 0\}$$

where $\mathbb{P}^2_{\mathbb{C}} = (\mathbb{C}^3 - \{0\})/\mathbb{C}^*$ is the complex projective plane, and F^h is the homogenization of F (the smallest degree homogeneous polynomial such that $F^h(x, y, 1) = F(x, y)$). $\overline{X}$ is a projective algebraic curve. In general, $\overline{X}$ could be singular, but let us ignore this for now, and suppose that $X = \overline{X}$ is nonsingular.[1] $\mathbb{C}^3 - \{0\}$ will inherit a Hausdorff topology from $\mathbb{C}^3$. Taking the quotient topology makes $\mathbb{P}^2$ into a compact Hausdorff space (which is very different from the Zariski topology). We give X the induced topology, then it is compact and Hausdorff.

1.2 Riemann surfaces

It is time to give a rigorous definition.[2] A *Riemann surface* or a *one dimensional complex manifold* or a *nonsingular complex curve* (these terms are interchangeable) consists of the following data:

1. A metrizable topological space X.
2. An open cover $\{U_i\}$ of X.
3. A collection of homeomorphisms $\phi_i : U_i \to \Delta$ to a disk, such that $\phi_i \circ \phi_j^{-1}$ are holomorphic.

The sets U_i are called coordinate disks or charts, and the composition $z \circ \phi_i$ (which is usually just written as z_i or just z_i) is called a local coordinate. We call $x_i = Re\, z_i, y_i = Im\, z_i$ the real coordinates. We can define higher dimensional complex manifolds and C^∞ manifolds in the same way, with Δ replaced by a ball in $\mathbb{R}^n$ or a product of disks in $\mathbb{C}^n$. Condition 3 is that $\phi_i \circ \phi_j^{-1}$ is either C^∞ or holomorphic. A function of several variables is holomorphic if it is continuous and holomorphic in each variable, when the other variables are fixed.

Let us consider some examples. Obviously:

Example 1.2.1. *Any open subset of $\mathbb{C}$ gives a Riemann surface.*

The first nontrivial example that we learn in basic complex analysis is

Example 1.2.2. *The Riemann sphere S^2 consists of the sphere with $U_0 = S^2 - (\text{south pole})$ and $U_\infty = S^2 - (\text{north pole})$. The function ϕ_0 is given by stereographic projection. If $z = z_0$ is the coordinate on U_0, the coordinate z_∞ on U_∞ satisfies $z_\infty = z^{-1}$, when it make sense. Note that algebraic geometers prefer to think of this as $\mathbb{P}^1_{\mathbb{C}}$.*

[1] For people who know what this means, in general one can always blow up $\overline{X}$ obtain a nonsingular curve X.

[2] As far as I can tell, this goes back to Weyl. His 1913 book on Riemann surfaces gave the first completely rigourous treatment of this topic.

Lemma 1.2.3. *A nonsingular affine algebraic curve has the structure of a Riemann surface in a natural way.*

Proof. Let $f(x,y)$ be a polynomial such that ∇f does not vanish on $X = V(f)$. Let $(x_i, y_i) \in X$ be a point such that $\frac{\partial f}{\partial x}(x_i, y_i) \neq 0$ (resp. $\frac{\partial f}{\partial y}(x_i, y_i) \neq 0$) , then the holomorphic implicit function [Griffiths-Harris, p 19] says that there exits open sets $U_i = \{|x - x_i| < \epsilon_i\}$, $V = \{|y - y_i| < \delta_i\}$ and a holomorphic function $g : U_i \to V_i$ such that $X \cap U_i \times V_i = \{(x,y) \mid y = g(x)\}$ (resp. with roles of x and y reversed). Then the collection $\{X \cap U_i \times V_i\}$ gives an open cover with ϕ_i given by projection to U_i. □

Lemma 1.2.4. *A nonsingular algebraic curve in the projective plane has the structure of a Riemann surface in a natural way.*

Proof. Let $X \subset \mathbb{P}^2$ be a nonsingular curve. Let $U_i = \{[x_0, x_1, x_2] \mid x_i \neq 0\}$, where x_i are homogeneous coordinates. Then $U_i \cong \mathbb{C}^2$ where for example when $i = 0$, the bijection is given by $[x_0, x_1, x_2] \mapsto (x_1/x_0, x_2/x_0)$. Under this bijection, X maps to a nonsingular affine curve. We can now apply the previous lemma. □

Example 1.2.5. *Let $L \subset \mathbb{C}$ be a lattice, which means $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$, where α and β are $\mathbb{R}$-linearly independent, e.g. $\alpha = 1, \beta = i$. Consider $X = \mathbb{C}/L$. Topologically, this is a torus. Choose a disk Δ centered at 0 and contained in the parallelogram with corners $\pm\alpha/2, \pm\beta/2$. For any $p \in X$, lift it to $\tilde{p} \in \mathbb{C}$, and let U be the image of $\Delta + \tilde{p}$. This gives a coordinate disk,*

We will see other examples that later. Given a Riemann surface X and an open set $U \subseteq X$. A function $f : U \to \mathbb{C}, \mathbb{R}$ is holomorphic (resp. C^∞) if its restriction to any coordinate disk is given by a holomorphic (resp. C^∞) function of the local coordinate z (coordinates x, y).

Theorem 1.2.6. *If X is a compact connected Riemann surface, then a holomorphic function on it is constant.*

Proof. Let $f : X \to \mathbb{C}$ be holomorphic. Since X is compact, $|f|$ must attain a maximum somewhere, say $p \in X$. Let $c = f(p)$ and $Z = \{q \in X \mid f(q) = c\}$. Then Z is closed. Choose $q \in Z$, and choose a coordinate disk $\Delta \subset X$ containing q. Since $|f|$ has a maximal value at an interior point of Δ, the maximum principle from complex analysis tells us that $f|_\Delta$ must be constant. Therefore $Z \supset \Delta$, which implies that it is open. Since Z is open and closed, and X is connected, we must have $Z = X$. □

A continuous map $f : X \to Y$ between Riemann surfaces is called *holomorphic* if it can be expressed as a holomorphic function of local coordinates. More precisely, for any $p \in X$, choose coordinates $\phi_p : U \xrightarrow{\sim} V \subset \mathbb{C}$ and $\psi_q : U' \xrightarrow{\sim} V' \subset \mathbb{C}$ at p and $q = f(p)$, then $\psi_q \circ f \circ \phi_p^{-1}$ should be holomorphic. A map f is an isomorphism if it is bijective and both f and f^{-1} are holomorphic. Clearly $f : X \to \mathbb{C}$ is holomorphic in the current sense if it $f \in \mathcal{O}(X)$. A holomorphic function $f : X \to \mathbb{P}^1$ is called a meromorphic function on X. The restriction of a meromorphic function to a coordinate disk is a meromorphic in the usual sense.

Theorem 1.2.7. *A nonconstant holomorphic between compact connected Riemann surfaces is surjective.*

Proof. Given $f : X \to Y$ as above, $f(X)$ is closed since X is compact. On the other hand, $f(X)$ is also open by basic complex analysis. So $f(X) = Y$. □

1.3 A little sheaf theory

To completely check that something is a manifold or Riemann surface using the previous standard definition can get a little tedious. This means that details tend to get omitted in practice. We give an alternative definition based on sheaf theory which is sometimes easier to use and pretty natural for an algebraic geometer. But to avoid spending a lot of time on foundations, we will just talk about sheaves of functions.[3] Given a topological space X and a set S, a *presheaf* of S-valued functions on X, is a collection $\mathcal{F}(U)$ of functions $U \to S$, for open sets U, such that $f|_V \in \mathcal{F}(V)$ whenever $f \in \mathcal{F}(U)$ and $V \subset U$. We say that $\mathcal{F}$ is a *sheaf* if $f : U \to S$ lies in $\mathcal{F}(U)$ if $f|_{U_i} \in \mathcal{F}(U_i)$ for any open cover of U. If S has an abelian group or ring, we say that $\mathcal{F}$ is sheaf of abelian groups or rings if $\mathcal{F}(U)$ is an abelian group or ring under pointwise operations. If $\mathcal{F}$ is a sheaf on X, then for any open $U \subset X$, the restriction $\mathcal{F}|_U(V) = \mathcal{F}(V)$, for $V \subseteq U$, gives a sheaf on U.

The following examples are sheaves of rings of $\mathbb{C}$-valued functions:

1. The collection of all continuous functions $C(U)$ on a space X.
2. The collection of C^∞ functions $C^\infty(U)$ on a C^∞ manifold X.
3. The collection of all holomorphic functions $\mathcal{O}(U)$ on a complex manifold X.

The following is a presheaf but not a sheaf.

1. The collection of constant functions on $\mathbb{C}$ (or almost any space X).

Let k be a field. By a ringed space over k, we mean a pair $(X, \mathcal{F})$ consisting of a topological space X and a sheaf of algebras of k-valued functions on it. A morphism $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ between k-ringed spaces is a continuous map such that $g \in \mathcal{G}(U)$ implies that $f^*g := g \circ f \in \mathcal{F}(f^{-1}U)$. This is called an isomorphism if f^{-1} exists and is also a morphism.

We will be interested in the following examples:

[3] In fact, this is not a real restriction, because any sheaf is isomorphic to a sheaf of functions to an appropriate target.

1. (X, C^∞_X) is an $\mathbb{R}$-ringed space, where X is (real) C^∞ manifold. A morphism $f : (X, C^\infty_X) \to (Y, C^\infty_Y)$ is the same thing as C^∞ map of manifolds. An isomorphism is the same thing as a diffeomorphism.

2. $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-ringed space, where X is a complex manifold. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the same thing as holomorphic map of manifolds (and in particular Riemann surfaces). An isomorphism is the same thing as biholomorphism (we will just use the word isomorphism).

Finally, we have the following alternative definition of Riemann surfaces etc.

Proposition 1.3.1. *A Riemann surface is the same thing a ringed space $(X, \mathcal{O}_X)$ over $\mathbb{C}$, such that X is metrizable and such that it is locally isomorphic to $(\Delta, \mathcal{O}_\Delta)$, where $\Delta \subset \mathbb{C}$ is a the unit disk. More precisely, there exists an open cover $\{U_i\}$ such that $(U_i, \mathcal{O}_X|_{U_i}) \cong (\Delta, \mathcal{O}_\Delta)$. Similar statements hold for C^∞ and complex manifolds.*

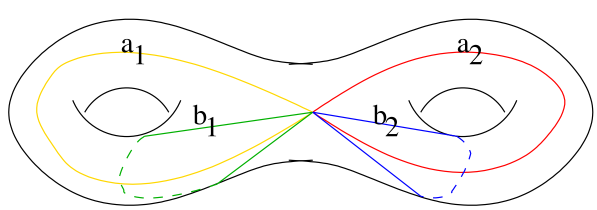
Exercise 1.3.2. *Let Γ be a subgroup of the automorphism group of Riemann surface X, and assume that the action is free and properly discontinuous (this means every point p has a closed nbhd K such that $\gamma(K) \cap K = \emptyset$ for $\gamma \neq 1$). Let $Y = X/\Gamma$ with quotient topology, let $\pi : X \to Y$ denote the projection,and let $\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}U)^\Gamma$ the ring of invariant functions. With the help of the last proposition, show that $(Y, \mathcal{O}_Y)$ is a Riemann surface.*

1.4 Topological invariants

If X is a Riemann surface, then the connected components are also Riemann surfaces. So we may usually restrict our attention to the connected surfaces (and this assumption is not always stated explicitly).

Theorem 1.4.1 (Topological classification)**.** *A compact connected Riemann surface is homeomorphic to a sphere with g handles. The number $g \in \mathbb{N}$ is called the genus.*

For example,



has genus 2. This is really a theorem topology about compact oriented 2-manifolds. The proof can be found in several places, such as Seifert and Threllfal's classic "Lectures in topology", which originally written in the 1930's.

Although the genus is the key topological invariant, it is not the only one. The other invariant we want to discuss is the first Betti number. We first define from the de Rham point of view. In calculus, given C^∞ functions f, g on an open set $U \subset \mathbb{R}^2$, we define an expression

$$\omega = f(x,y)dx + g(x,y)dy$$

to be a C^∞ differential form of degree 1 or simply 1-form on U. Let $\mathcal{E}^1(U)$ denote the vector space of 1-forms on U. A basic question is when can we find a C^∞ function h such that

$$\omega = dh := \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$$

A necessary condition is that the 2-form

$$d\omega := \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = 0$$

One says that ω is closed if the last condition holds, and exact if $\omega = df$. A closed form need not be exact as can be seen using $U = \mathbb{R}^2 - \{0\}$ and ω equals "$d\theta$" (note that θ is not a true function on U). We measure the failure by introducing the first de Rham cohomology

$$H^1_{dR}(U, \mathbb{R}) = \frac{\{\omega \in \mathcal{E}^1(U) \mid d\omega = 0\}}{\{df \mid f \in C^\infty(U)\}}$$

The first Betti number $b_1(U)$ is

$$\dim_{\mathbb{R}} H^1_{dR}(U, \mathbb{R})$$

We extend these definitions to a Riemann surface X as follows: A C^∞ 1-form ω on X is an assignment of a 1-form in the above sense $\omega_i = f_i dx_i + g_i dy_i$, for every system of real coordinates. These are required to be compatible with coordinate changes in the sense that

$$\omega_j = f_i \left(\frac{\partial x_i}{\partial x_j}dx_j + \frac{\partial x_i}{\partial y_j}dy_j\right) + g_i \left(\frac{\partial y_i}{\partial x_j}dx_j + \frac{\partial y_i}{\partial y_j}dy_j\right)$$

(This is pretty much the classical approach. There are coordinate free approaches which, however, take more time to set up.) The other notions extend in the same way, and we can define $H^1_{dR}(X, \mathbb{R})$ and the first Betti number as above. Letting $\mathcal{E}^0(X) = C^\infty(X)$ and $\mathcal{E}^2(X)$ denote the space of 0-forms and 2-forms. We define the other de Rham cohomologies by

$$H^0_{dR}(X, \mathbb{R}) = \{f \in \mathcal{E}^0(X) \mid df = 0\}$$

$$H^2_{dR}(X, \mathbb{R}) = \{\omega \in \mathcal{E}^0(X) \mid d\omega = 0\}$$

It is easy to see that $df = 0$ implies that f is constant. Therefore

$$H^0_{dR}(X, \mathbb{R}) \cong \mathbb{R}$$

It is also true that

$$H^2_{dR}(X, \mathbb{R}) \cong \mathbb{R}$$

but this is harder to see. Thus the dimensions

$$b_0(X) = b_2(X) = 1$$

The Euler characteristic of any topological space (for which b_i is defined and $\sum b_i < \infty$) is given by

$$e(T) = \sum (-1)^i b_i(T)$$

Therefore

$$e(X) = b_0(X) - b_1(X) + b_2(X) = 2 - b_1(X)$$

One nice feature of the Euler characteristic is the following "inclusion-exclusion" property

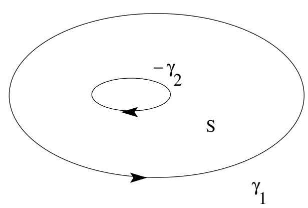
Theorem 1.4.2. *If $T = U \cup V$ is a union of open sets,*

$$e(T) = e(U) + e(V) - e(U \cap V)$$

Proof. This follows from the Mayer-Vietoris sequence and the additivity of dim for exact sequences. □

Exercise 1.4.3. *Use the above theorem to show that $e(X) = 2 - 2g$ when X is a genus g compact surface. Conclude that $b_1(X) = 2g$.*

There is a dual point of view, which is more geometric. Roughly speaking the first homology $H_1(X, \mathbb{Z})$ has generators consisting of closed oriented C^∞ paths, or loops, in X. Two loops γ_1, γ_2 define the same element of $H_1(X, \mathbb{Z})$ if there exists subsurface $S \subset X$ whose boundary is $\gamma_1 - \gamma_2$.



For a more complete treatment, see for example Hatcher's Algebraic Topology. Given a closed 1-form ω, Stokes' theorem shows that

$$\gamma \mapsto \int_\gamma \omega$$

gives a well defined element of

$$Hom(H_1(X, \mathbb{Z}), \mathbb{R})$$

Theorem 1.4.4 (de Rham)**.** *The above map gives an isomorphism*

$$H^1_{dR}(X, \mathbb{R}) \cong Hom(H_1(X, \mathbb{Z}), \mathbb{R})$$

Therefore $rank H_1(X, \mathbb{Z}) = 2g$. In fact, it is known that $H_1(X, \mathbb{Z})$ is torsion free, therefore it is isomorphic to $\mathbb{Z}^{2g}$. In the genus 2 example depicted above, the loops a_1, a_2, b_1, b_2 denote the generators.

1.5 Elliptic curves

Let $L \subset \mathbb{C}$ be a lattice, i.e. subgroup spanned by two $\mathbb{R}$-linearly independent numbers ω_i. The torus $E = \mathbb{C}/L$ is called an *elliptic curve*. We will see below that such a curve can be realized as cubic curve in the plane. Since $E \cong \mathbb{C}/\omega_1^{-1}L$, there is no loss in assuming that $\omega_1 = 1$, and that $Im(\omega_2) > 0$ (replace ω_2 by $-\omega_2$ if necessary). A translate of a parellogram having corners $0, 1, \omega_2, 1 + \omega_2$ will be referred to as a fundamental parallelogram.

Now consider complex function theory on E. Any function on E can be pulled back to a function f on $\mathbb{C}$ such that

$$f(z + \lambda) = f(z), \quad \lambda \in L \tag{1.2}$$

A meromorphic function satisfying this is called a doubly periodic function or an *elliptic function* with respect to L.

Proposition 1.5.1. *Any holomorphic elliptic function is constant.*

First proof. An holomorphic elliptic function is a holomorphic function on E. Earlier we proved that holomorphic functions on compact Riemann surfaces are constant. □

Second proof. A holomorphic elliptic function is a bounded entire function. This is constant by Liouville's theorem. □

So to get interesting elliptic functions, we must have poles. For example:

Theorem 1.5.2. *The Weirerstrass $\wp$-function*

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in L - \{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]$$

is an even elliptic function with double poles at points of L and no other singularities.

First we need the following, which can be proved using elementary analysis.

Lemma 1.5.3. *If $k > 2$, the series, called an Eisenstein series,*

$$\sum_{\lambda \in L - \{0\}} \frac{1}{\lambda^k}$$

converges absolutely.

Proof of theorem. Some elementary manipulations lead to an inequality

$$\left| \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right| = \left| \frac{2\lambda z - z^2}{\lambda^2 (z-\lambda)^2} \right| \leq \frac{const}{|\lambda|^3}$$

when z lies in a bounded subset of $\mathbb{C} - L$. Therefore we see that the series for $\wp(z)$ converges uniformly on compact sets away from L by the previous lemma.

The series shows that $\wp(-z) = \wp(z)$, so it is even. By uniform convergence, we can differentiate term by term to get

$$\wp'(z) = -2\sum_{\lambda\in L} \frac{1}{(z-\lambda)^3}$$

This is clearly elliptic. Therefore

$$\wp(z+\lambda) = \wp(z) + c(\lambda)$$

for some $c(\lambda)$ which is independent of z. Choosing $z = -\lambda/2$, and using the evenness of $\wp(z)$ shows that $c(\lambda) = 0$. This implies that $\wp(z)$ is elliptic. Clearly it has a double pole at 0. and therefore at all points of L.

□

We will need the following below.

Lemma 1.5.4. *Let $f(z)$ be a nonzero elliptic function, and P a fundamental parallelogram such that the poles of $f(z)$ do not lie on the boundary of P. Then the sum of orders of f within P equals the sum of the orders of the poles.*

Proof. By complex analysis the difference between the above orders is

$$\frac{1}{2\pi i}\int_{\partial P} \frac{f'(z)}{f(z)} dz$$

Double periodicity of f implies that the integral along opposite sides of P cancel.

□

Exercise 1.5.5. *Use the fact that $\wp'(z)$ is an odd elliptic function to prove that $S = \{\frac{1}{2}, \frac{\omega_2}{2}, \frac{1+\omega_2}{2}\}$ are zeros of this function. Choose a fundamental parallelogram P' with 3 corners given by the above points. Let $P = P' + \epsilon(1+\omega_2)$ with $\epsilon > 0$ small, so that the points S lie in the interior of P. Use the lemma to show that the zeros of $\wp'(z)$ in P are exactly the points in S.*

The next step is to relate this to algebraic geometry by embedding E into projective space. Denote the image of 0 in $\mathbb{C}$ by 0 as well.

Theorem 1.5.6.

(a) $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ for the appropriate choice of constants g_i.

(b) The affine and projective algebraic curves defined by

$$y^2 = 4x^3 - g_2x - g_3$$

and

$$zy^2 = 4x^3 - g_2xz^2 - g_3z^3$$

are nonsingular.

(c) *The map* $z \mapsto (\wp(z), \wp'(z)) \in \mathbb{C}^2$*, gives a well defined map of* E *minus* 0 *to* $\mathbb{C}^2$*. This gives an isomorphism between* $E - \{0\}$ *(resp.* E*) and the affine (projective) cubic defined in (b).*

Sketch. See [Silverman, Arithmetic of elliptic curves, chap VI] for complete details. The idea for (a) is to choose the constants so that the difference $(\wp')^2 - 4\wp^3 - g_2\wp - g_3$ vanishes at 0. But then it is elliptic with no poles, so it is constant. Therefore it vanishes everywhere.

Using the previous exercise and (a), we can see that $4x^3 - g_2x - g_3$ has 3 distinct roots. It is easy to see using that the curves defined in (b) are nonsingular.

Let $\phi(z) = (\wp(z), \wp'(z))$. Clearly this factors through $\Phi : E - \{0\} \to \mathbb{C}^2$ and the image lies within the affine cubic given in (b). We will be content to prove that Φ is injective. Suppose not. Then $\phi(z_1) = \phi(z_2)$ for $z_1 - z_2 \notin L$. Let P be a fundamental parallelogram which is symmetric about 0. After translating P slightly, we can assume without loss of generality that $\pm z_1, z_2$ lie in the interior of P. The function $f(z) = \wp(z) - \wp(z_1)$ is even, so it must vanish at $\pm z_1$ and z_2. Since $f(z)$ has a double pole at 0 and not other poles in P, we can conclude by the previous lemma that $f(z)$ can have at most 2 zeros. This forces $z_2 = -z_1$. Since $\wp'(z)$ is an odd function, $\wp'(z_2) = -\wp'(z_1)$. If $\wp'(z_1) \neq 0$, then $\phi(z_1) \neq \phi(z_2)$, which is a contradiction. Therefore $\wp'(z_1) = 0$, which implies that z_1 is a double zero of f. Therefore $z_2 = z_1$. In a nbhd of 0, $\phi(z) = [z^3\wp(z), z^3\wp'(z), z^3] \in \mathbb{P}^2$. Since $\wp'(z)$ has a triple pole at 0, this shows that we have a holomorphic extension with $\phi(0) = [0, 1, 0]$. This gives an injective holomorphic map Φ from E to the projective cubic in (b). Since Φ is nonconstant, it also surjective. This almost proves that it is an isomorphism. To make sure, we need to check that the derivative everywhere nonzero. See Silverman for this. □

In summary, an elliptic curve really is an algebraic curve, which can be realized as a plane cubic. In the purely algebraic theory, which works over any field, one starts with the latter.

1.6 Jacobi's Theta function

The alternative approach to getting interesting holomorphic functions on a lattice is to relax the periodicity (1.2). This leads to the theory of theta functions. The higher dimensional analogue will play an important role later. Basically, we want holomorphic functions that satisfy

$$f(z + \lambda) = (\text{some factor})f(z)$$

which we refer to as quasi-periodicity with respect to $L = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau = \omega_2$ in the upper half plane. Similar ideas are in the theory of automorphic forms. We can obtain elliptic functions by taking ratios of two such functions with the same factors. To make it more precise, we want

$$f(z + \lambda) = \phi_\lambda(z)f(z) \tag{1.3}$$

where $\phi_\lambda(z)$ is a nowhere zero entire function. To guarantee nonzero solutions, we require some compatibility conditions

$$f(z+(\lambda_1+\lambda_2)) = \phi_{\lambda_1+\lambda_2}(z)f(z)$$

$$f((z+\lambda_1)+\lambda_2) = \phi_{\lambda_2}(z+\lambda_2)\phi_{\lambda_1}(z)f(z)$$

which suggests that we should impose

$$\phi_{\lambda_1+\lambda_2}(z) = \phi_{\lambda_2}(z+\lambda_1)\phi_{\lambda_1}(z)$$

This is called the 1-cocycle identity. As it turns out, there is a cheap way to get solutions, choose a nowhere zero function $g(z)$ and let $\phi_\lambda(z) = g(z+\lambda)/g(z)$ such as cocycle is called a coboundary. From the point of view of constructing interesting solutions of (1.3), it is not very good. Any solution would be a constant multiple of $g(z)$. Taking ratios of two of these functions would result in a constant.

The problem of constructing cocycles which are not coboundaries can be solved using the machinery of group cohomology. The set of cocycles modulo coboundaries forms the cohomology group $H^1(L, \mathcal{O}(\mathbb{C})^*)$. There is a connecting homomorphism[4]

$$c_1 : H^1(L, \mathcal{O}(\mathbb{C})^*) \to H^2(L, \mathbb{Z}) \cong \wedge^2 L^*$$

to the space of alternating integer valued forms on L. Given a cocycle ϕ_λ, to show that it is not a coboundary is enough to show that the image of c_1 is nonzero. Fortunately this can be done explicitly. Since ϕ_λ is entire and nowhere 0, we can take a global logarithm $\psi_\lambda(z) = \log \phi_\lambda(z)$. Then

$$F(\lambda_1, \lambda_2) = \frac{1}{2\pi i}[\psi_{\lambda_1+\lambda_1}(z) - \psi_{\lambda_2}(z+\lambda_2) - \psi_{\lambda_1}(z)] \in \mathbb{Z}$$

gives an integer valued function such that

$$c_1(\phi_\bullet)(\lambda_1, \lambda_2) = F(\lambda_1, \lambda_2) - F(\lambda_2, \lambda_1)$$

One can check that the exponential of

$$\psi_{n\tau+m}(z) = -n^2\pi i\tau + 2\pi inz$$

gives a cocycle whose image under c_1 is nonzero. With this choice, we can find an explicit solution to (1.3). The Jacobi θ-function is given by the Fourier series

$$\theta(z) = \sum_{n\in\mathbb{Z}} \exp(\pi in^2\tau + 2\pi inz) = \sum_{n\in\mathbb{Z}} \exp(\pi in^2\tau)\exp(2\pi inz)$$

[4]As the notation suggests, it is a version of the first Chern class. People familiar with group cohomology can verify the last isomorphism for $H^2(L, \mathbb{Z})$ using the Koszul resolution, otherwise take it as a blackbox.

Writing $\tau = x + iy$, with $y > 0$, shows that on a compact subset of the z-plane the terms are bounded by $O(e^{-n^2 y})$. So uniform convergence on compact sets is guaranteed. This is clearly periodic

$$\theta(z+1) = \theta(z)$$

In addition it satifies the function equation

$$\begin{aligned}\theta(z+\tau) &= \sum \exp(\pi i n^2\tau + 2\pi i n(z+\tau)) \\ &= \sum \exp(\pi i (n+1)^2\tau - \pi i\tau + 2\pi i n z) \\ &= \exp(-\pi i\tau - 2\pi i z)\theta(z)\end{aligned}$$

and more generally

$$\theta(z + n\tau + m) = \exp(\psi_{n\tau+m}(z))\theta(z)$$

We can get a larger supply of quasiperiodic functions by translating. Given a rational number b, define

$$\theta_{0,b}(z) = \theta(z+b)$$

Then

$$\theta_{0,b}(z+1) = \theta_{0,b}(z), \quad \theta_{0,b}(z+\tau) = \exp(-\pi i\tau - 2\pi i z - 2\pi i b)\theta(z)$$

We can construct elliptic functions by taking ratios: $\theta_{0,b}(Nz)/\theta_{0,b'}(Nz)$ is a (generally nontrivial) elliptic function when $b, b' \in \frac{1}{N}\mathbb{Z}$. More generally given rational numbers $a, b \in \frac{1}{N}\mathbb{Z}$, we can form the theta functions with characteristics

$$\theta_{a,b}(z) = \exp(\pi i a^2\tau + 2\pi a(z+b))\theta(z + a\tau + b) \tag{1.4}$$

Fix $N \geq 1$, and let V_N denote the set of linear combinations of these functions.

Lemma 1.6.1. *Given nonzero $f \in V_N$, it has exactly N^2 zeros in the parallelogram with vertices $0, N, N\tau, N+\tau$.*

Sketch. Complex analysis tells us that the number of zeros is given by the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)dz}{f(z)}$$

over the boundary of the parallelogram. This can be evaluated to N^2 using the identities $f(z+N) = f(z)$, $f(z+N\tau) = Const.\exp(-2\pi i N z)f(z)$ following from (1.4). □

These can be used to construct a projective embedding different from the previous.

Theorem 1.6.2. *Choose an integer $N > 1$ and the collection of all θ_{a_i,b_i}, as (a_i, b_i) runs through representatives of $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$. The the map of $\mathbb{C}/L$ into $\mathbb{P}^{N^2-1}$ by $z \mapsto [\theta_{a_i,b_i}(z)]$ is an embedding.*

Sketch. Suppose that this is not an embedding. Say that $f(z_1) = f(z_1')$ for some $z_1 \neq z_1'$ in $\mathbb{C}/L$ and all $f \in V_N$. By translation by $(a\tau + b)/N$ for $a, b \in \frac{1}{N}\mathbb{Z}$, we can find another such pair z_2, z_2' with this property. Since $\dim V_N = N^2$, we can find additional points $z_2, \ldots z_{N^2-3}$, distinct in $\mathbb{C}/NL$, so that

$$f(z_1) = f(z_2) = f(z_3) = \ldots f(z_{N^2-3}) = 0$$

for some $f \in V_N - \{0\}$. Notice that we are forced to also have $f(z_1') = f(z_2') = 0$ which means that f has at least $N^2 + 1$ zeros which contradicts the lemma.

Further details can be found in [Mumford, Lectures on Theta I] □

Note that the smallest such embedding lands in $\mathbb{P}^3$, so it is different from what we obtained before.