Chapter 2

Hodge theory

2.1 Cauchy-Riemann operator

Let $U \subset \mathbb{C}$ be an open set Let $C^\infty(U)$ (resp. $C^\infty_\mathbb{R}(U)$) denote the space of complex (resp. real) valued functions. Similarly, we work with complex valued differential forms, where $\mathcal{E}^1(U)$ (resp. $\mathcal{E}^1_\mathbb{R}(U)$) denotes the space of complex (resp. real) valued 1-forms. Note that $\mathcal{E}^1(U)$ is a module over $C^\infty(U)$. If

$$z = x + iy$$

as usual, and introduce complex valued differential forms

$$dz = dx + idy,\ d\bar{z} = dx - idy$$

Therefore

$$dx = \frac{1}{2}(dz + d\bar{z})$$

$$dy = \frac{1}{2i}(dz - d\bar{z})$$

Given a C^∞ function $f : U \to \mathbb{C}$, the total differential

$$df = f_x dx + f_y dy = \frac{1}{2}(f_x - if_y)dz + \frac{1}{2}(f_x + if_y)d\bar{z}$$

This suggests that we should introduce the operators

$$\partial f = \frac{1}{2}(f_x - if_y)dz$$

$$\bar{\partial} f = \frac{1}{2}(f_x + if_y)d\bar{z}$$

so that

$$d = \partial + \bar{\partial}$$

If we set $u = Re\, f, v = Im\, f$, then

$$\bar\partial f = \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)]d\bar z$$

This makes it clear that the condition $\bar\partial f = 0$ is equivalent to the Cauchy-Riemann equations. Therefore

Lemma 2.1.1. *$f \in C^\infty(U)$ is holomorphic if and only if $\bar\partial f = 0$.*

We let $\mathcal{E}^{10}(U) \subset \mathcal{E}^1(U)$ (resp. $\mathcal{E}^{01}(U) \subset \mathcal{E}^1(U)$) be the submodule spanned by dz (resp. $d\bar z$). We call these forms of type $(1,0)$ or $(0,1)$. We have

$$\mathcal{E}^1(U) = \mathcal{E}^{10}(U) \oplus \mathcal{E}^{01}(U)$$

and ∂ (resp. $\bar\partial$) is just d followed by projection to these submodules.

We now want to show that of this make sense on a Riemann surface X. Given two overlapping coordinate disks U and V with local coordinates z and ζ, we see that ζ is a holomorphic function of z and visa versa. Therefore

$$d\zeta = \partial\zeta = \frac{\partial\zeta}{\partial z}dz$$

$$dz = \partial z = \frac{\partial z}{\partial\zeta}d\zeta$$

Therefore

$$\mathcal{E}^{10}(U\cap V) = C^\infty(U\cap V)dz = C^\infty(U\cap V)d\zeta$$

We can now define $\mathcal{E}^{10}(X) \subset \mathcal{E}^1(X)$ to be the space of 1-forms whose restriction to any coordinate disk U_i lies $\mathcal{E}^{10}(U_i)$. The previous equality shows that this is well defined. We define $\mathcal{E}^{01}(X)$ to be the space of complex conjugates of $(1,0)$-forms. We can see that any form in $\mathcal{E}^1(X)$ has a unique decomposition into a sum of $(1,0)$-form and $(0,1)$-form. Therefore

$$\mathcal{E}^1(X) = \mathcal{E}^{10}(X) \oplus \mathcal{E}^{01}(X)$$

We define ∂f (resp. $\bar\partial f$) to be the projection of df to the first (resp. second) factor. A $(1,0)$-form is called *holomorphic* if its restriction to any coordinate disk with coordinate z is $f(z)dz$ with f holomorphic. We let $\Omega^1(X)$ denote the space of holomorphic 1-forms.

2.2 Harmonic forms

Fix a compact (connected) Riemann surface X. Let us suppose that the genus is g. As before $C^\infty(X)$ and $\mathcal{E}^p(X)$ will now denote the spaces of complex valued C^∞ functions and complex valued forms. We these conventions, we can define complex valued de Rham cohomology as before

$$H^1_{dR}(X,\mathbb{C}) = \frac{\{\alpha \in \mathcal{E}^1(X) \mid d\alpha = 0\}}{\{df \mid f \in C^\infty(X)\}}$$

This is isomorphic to $H^1_{dR}(X,\mathbb{R})\otimes\mathbb{C}=\mathbb{C}^{2g}$. Note the formula and similar ones appear more uniform, if we set

$$\mathcal{E}^0_X=\mathcal{E}^{00}_X=\mathbb{C}_X$$

We note that Riemann surfaces have a canonical orientation: if x,y are real and imaginary parts of a complex coordinate z, then $dx\wedge dy$ is positively oriented. The orientation allows us to integrate two forms on X. Given $\alpha,\beta\in\mathcal{E}^1(X)$, define

$$(\alpha,\beta)=\int_X\alpha\wedge\beta$$

Stokes' theorem and properties of the wedge product shows that this gives a well defined skew symmetric pairing

$$(\ ,\ ):H^1_{dR}(X,\mathbb{C})\times H^1_{dR}(X,\mathbb{C})\to\mathbb{C}$$

For people familiar with it, this is dual to the (complexified) intersection pairing on $H_1(X,\mathbb{Z})$

An element of de Rham cohomology is really an equivalence class. *Does such a class have a distinguished representative?* The answer will turn out to be yes. To describe it, let us introduce a $C^\infty(X)$-linear operation called the Hodge star given locally by $*dx=dy$, $*dy=-dx$. This is amounts to multiplication by i in the cotangent planes, so it is globally well defined operation. We have the following basic properties

Lemma 2.2.1. *$\mathcal{E}^1(X)$ has an inner product given by*

$$\langle\alpha,\beta\rangle=(\alpha,*\bar\beta)=\int_X\alpha\wedge*\bar\beta$$

Proof. One can see that

$$(fdx+gdy)\wedge*\overline{(hdx+kdy)}=(f\bar h+g\bar k)dx\wedge dy$$

$$(fdx+gdy)\wedge*\overline{(fdx+gdy)}=(|f|^2+|g|^2)dx\wedge dy$$

This implies the basic properties including positive definiteness. □

Corollary 2.2.2 (Poincaré duality)**.** *The bilinear form* (,) *is nondegenerate.*

Remark 2.2.3. *The topological form of Poincaré duality gives the stronger result that the intersection pairing on $H_1(X,\mathbb{Z})$ is unimodular. This means that H_1 has a basis, called a symplectic basis, such that the pairing is represented by*

$$\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$$

We will use this later on.

Definition 2.2.4. *We define a* 1*-form* α *to be co-closed if* $d(*\alpha) = 0$. *It is harmonic if it is both closed and co-closed, i.e.* $d\alpha = d(*\alpha) = 0$. *A form is called co-exact if it equals* $*df$.

The reason for the name will be explained later on. The basic properties are given by:

Proposition 2.2.5.

(a) A harmonic 1*-form is a sum of a* $(1,0)$ *harmonic form and* $(0,1)$ *harmonic form.*

(b) A $(1,0)$*-form is holomorphic if and only it is closed if and only if is harmonic.*

(c) A $(0,1)$*-form is harmonic if and only it is antiholomorphic i.e. its complex conjugate is holomorphic.*

(d) A 1*-form is co-closed (resp. closed) forms if and only if it is orthogonal the space of exact (resp. co-exact) forms. Therefore a* 1*-form is harmonic if and only if it is orthogonal to the direct spaces of*

Proof. If α is a harmonic 1-form, then $\alpha = \alpha' + \alpha''$, where $\alpha' = \frac{1}{2}(\alpha + i * \alpha)$ is a harmonic $(1,0)$-form and $\alpha'' = \frac{1}{2}(\alpha - i * \alpha)$ is a harmonic $(0,1)$-form.

If α is $(1,0)$, then $d\alpha = \bar{\partial}\alpha$. This implies the first half (b). For the second half, use the identity

$$*dz = *(dx + idy) = dy - idx = -idz$$

Finally, note that the harmoncity condition is invariant under conjugation, so the (c) follows from (b).

For (d), we first observe that integration by parts (essentially Stokes' theorem) implies

$$\langle df, \alpha \rangle = \int_X df \wedge *\bar{\alpha} = \int d(f * \bar{\alpha}) - \int_X f d * \bar{\alpha} = -\int_X f d * \bar{\alpha}$$

If α is co-closed, then it follows that $\langle df, \alpha \rangle = 0$. Conversely, suppose that $\langle df, \alpha \rangle = 0$ for all $f \in C^\infty(X)$. Let $d * \alpha = g(x,y) dx \wedge dy$ in a coordinate disk D. If $g(p) \neq 0$, we can choose f with support in D such that $f(x,y)g(x,y) \geq 0$ everywhere and strictly positive at p. Therefore $\int_X f d * \alpha > 0$, so we can conclude that $d * \alpha = 0$. A similar argument using

$$\langle \alpha, *df \rangle = \int_X \alpha \wedge * * d\bar{f} = -\int_X d(\bar{f}\alpha) + \int_X \bar{f} d\alpha = \int_X \bar{f} d\alpha \qquad (2.1)$$

shows that $d\alpha = 0$ if and only if α is orthogonal to co-closed forms.

□

Here is the key fact. We will say more about this in later on.

Theorem 2.2.6 (Hodge theorem). *Every de Rham cohomology class has a unique harmonic representative.*

Remark 2.2.7. *This statement is actually due Weyl, which Hodge generalized to higher dimensions.*

Corollary 2.2.8 (Hodge decomposition). *We have*

$$H^1_{dR}(X, \mathbb{C}) \cong \Omega^1(X) \oplus \overline{\Omega^1(X)}$$

Therefore $\dim \Omega^1(X) = g$.

Proof. By proposition 2.2.5, a harmonic 1 can be uniquely decomposed as a sum of holomorphic 1-form and the complex conjugate of a holomorphic 1-form. □

For reasons that will be explained later, one normally denotes $\Omega^1(X)$ by $H^0(X, \Omega^1_X)$ and this notation will be used below.

2.3 Proof of the Hodge theorem

First we explain the connection between harmonic forms and harmonic functions. Recall that C^∞ function f on an open subset of $\mathbb{R}^2$ is harmonic if it satisfies the Laplace equation

$$\Delta f := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0$$

Lemma 2.3.1. *A* 1*-form on a disk is harmonic if and only if it is given by* df*, where* f *is a harmonic function.*

Proof. Since a disk D is contractible, a 1-form on D is closed if and only if equals df for some f. The form is also co-closed when

$$d{*}df = \Delta f dx \wedge dy = 0$$

□

Recall that Green's identity from calculus implies that if f and η are both C^∞ and η vanishes near the boundary ∂D, then

$$\int_D f \Delta\eta \, dxdy = \int_D (\Delta f)\eta \, dxdy$$

Therefore if f is harmonic, then the first integral vanishes. Weyl's lemma is a converse statement.

Theorem 2.3.2 ("Weyl's lemma"). *Let* $D \subset \mathbb{C}$ *be an open disk. Let* $f \in L^2(D)$ *be such that*

$$\int_D f \Delta\eta \, dxdy = 0$$

for every compactly supported C^∞ *function* η*, then* f *is a* C^∞ *harmonic function.*

Proof. The proof is not difficult but it takes a few pages, so we refer to [Farkas-Kra, Riemann surfaces]. □

The proof of the Hodge theorem we give uses the method of orthogonal projection. The idea is to use a generalization of a fact from basic linear algebra that if $S \subset V$ is a subspace of a finite dimensional inner product space, then

$$V = S \oplus S^{\perp}$$

When V is infinite dimensional, this is no longer true unless V is a Hilbert space and S is closed. Thus we first need to complete everything to a Hilbert space in order to apply this. Let us denote by $L^2\mathcal{E}^1(X)$ the Hilbert space completion with of this space. Let $\mathcal{E}^1_{ex}(X), \mathcal{E}_{cl}(X), \mathcal{E}^1_{co}(X) \subset \mathcal{E}^1(X)$ denote the space of exact, closed and co-exact 1-forms i.e. the forms $*df$. Since these spaces are orthogonal, we get that the closure

$$\overline{\mathcal{E}^1_{ex}(X) + \mathcal{E}^1_{co}(X)} = \overline{\mathcal{E}^1_{ex}(X)} \oplus \overline{\mathcal{E}^1_{co}(X)}$$

in $L^2\mathcal{E}^1(X)$. Let H denote the orthogonal complement of the above space. Then we have an orthogonal decomposition

$$L^2\mathcal{E}^1(X) = H \oplus \overline{\mathcal{E}^1_{ex}(X)} \oplus \overline{\mathcal{E}^1_{co}(X)} \tag{2.2}$$

Lemma 2.3.3. *H consists of the space of harmonic C^∞ 1-forms.*

Proof. Given a C^∞ form α the orthogonality conditions defining H imply that H is harmonic. Given an element of $\alpha \in H$, its restriction to a coordinate disk D can be viewed as a differential form $\alpha|_D = pdx + qdy$ with L^2 coefficients. Let η be a C^∞ function with compact support on D. The orthogonality conditions imply that

$$\langle pdx + qdy, d\eta_x - *d\eta_y\rangle = 0$$

Expanding the left side yields

$$\int_D p\Delta\eta\, dxdy = 0$$

This implies that p is harmonic by Weyl's lemma. Similarly q is harmonic. Therefore α is C^∞, and consequently harmonic.

□

Lemma 2.3.4. $\mathcal{E}^1_{cl}(X) \cap \overline{\mathcal{E}^1_{ex}(X)} = \mathcal{E}^1_{ex}(X)$

Proof. If $\alpha \in \mathcal{E}^1_{ex}(X)$, and $\beta \in \mathcal{E}^1_{cl}(X)$, then Stokes' theorem implies that $(\alpha, \beta) = \langle \alpha, *\bar{\beta}\rangle = 0$. By continuity, this continues to hold for $\alpha \in \overline{\mathcal{E}^1_{ex}(X)}$. Now suppose that $\alpha \in \mathcal{E}^1_{cl}(X) \cap \overline{\mathcal{E}^1_{ex}(X)}$. We just showed that the cohomology class of α satisfies $(\alpha, \beta) = 0$ for any class $\beta \in H^1_{dR}(X, \mathbb{C})$. Therefore by Poincaré duality $[\alpha] = 0$. This implies that α is exact. □

Proof of the Hodge theorem. Let $\alpha \in \mathcal{E}^1_{cl}(X)$. Then using (2.2), we may decompose $\alpha = \beta + \gamma + \delta$, with $\alpha \in H$ etc. We claim that $\|\delta\|^2 = 0$. By continuity, it is enough to assume that $\delta = *df$. Then the orthogonality conditions plus (essentially) (2.1) shows that

$$\|\delta\|^2 = \langle \alpha, *df \rangle = \pm \int_X d(f\alpha) = 0$$

Therefore $\alpha = \beta + \gamma$. By lemma 2.3.3, β is harmonic. Therefore $\gamma = \alpha - \beta$ is in $\mathcal{E}^1_{cl}(X)$. By lemma 2.3.4, γ is exact. □

We won't give a proof, but it is possible to get a stronger result that the space of 1-forms decomposes as below.

Theorem 2.3.5 (Hodge theorem II)**.** *We have a decomposition*

$$\mathcal{E}^1(X) = H \oplus \mathcal{E}^1_{ex}(X) \oplus \mathcal{E}^1_{co}(X)$$

where H*,* $\mathcal{E}^1_{ex}(X)$ *and* $\mathcal{E}^1_{co}(X)$ *is the space of harmonic, exact and co-exact* 1*-forms respectively.*

2.4 Background on sheaf cohomology

We want to give a somewhat different interpretation of the Hodge decomposition, which will rely on the machinery of sheaf cohomology. We will mostly treat this machinery as a black box, or perhaps a dark grey box. More thorough treatments can be found in the books on algebraic geometry by Griffiths-Harris, Hartshorne, Voisin, and myself. Let us start with some definitions. Given a topological space X, a presheaf of abelian groups is a contravariant functor from the category $Open(X)$ of open sets of X, where morphisms are inclusions, to the categorry of abelian groups Ab. More concretely a presheaf is a collection of abelian groups $\mathcal{F}(U), U \in Open(X)$, with restrictions $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, when $V \subseteq U$, subject to appropriate compatibility conditions. Such a presheaf $\mathcal{F}$ is called a sheaf of abelian groups (henceforth just a sheaf) if for any open U with open cover $\{U_i\}$, any collection $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ is the restriction of a unique section $f \in \mathcal{F}(U)$. Let $Ab(X)$ denote the category of sheaves on X where a morphism is an additive natural transformation. This is an abelian category, so it comes with a natural notion of exact sequence. To spell it out, a sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is *exact* if for any $x \in X$, we can find an open nbhd U such that

$$0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$$

is exact in Ab, and for every $\gamma \in \mathcal{C}(U)$, after shrinking U, γ lies in the image of $\mathcal{B}(U)$. For the last part, it would suffice to assume that $\mathcal{B}(U) \to \mathcal{C}(U)$ is surjective, although the condition is a bit weaker.

Example 2.4.1. *Let X be a Riemann surface. Let $\mathbb{Z}_X$ denote the sheaf of locally constant $\mathbb{Z}$ functions on X, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ the sheaf of nowhere zero holomorphic functions viewed as a multiplicative group. We have an sequence*

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{e} \mathcal{O}_X^* \to 1$$

where the first map is the obvious one, and second sends $f \to \exp(2\pi i f)$. If U is a coordinate disk, then

$$0 \to \mathbb{Z}(U) \to \mathcal{O}_X(U) \xrightarrow{e} \mathcal{O}_X^*(U) \to 1$$

is exact. Therefore the above sequence of sheaves is exact. This called the exponential sequence.

Example 2.4.2. *Let X be a Riemann surface once again. Let $\mathbb{Z}_X$ denote the sheaf of locally constant $\mathbb{Z}$ functions on X,. Then the sequence*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \to 0$$

is exact. This follows from the exactness of

$$0 \to \mathbb{C}_X(U) \to \mathcal{O}_X(U) \xrightarrow{d} \Omega_X^1(U) \to 0$$

for a coordinate disk U.

Example 2.4.3. *Again let X be a Riemann surface. Then we have a sequence*

$$0 \to \mathcal{O}_X \to C_X^\infty \xrightarrow{\bar\partial} \mathcal{E}_X^{01} \to 0$$

which we claim is exact. It suffices to check the exactness of

$$0 \to \mathcal{O}_X(U) \to C_X^\infty(U) \xrightarrow{\bar\partial} \mathcal{E}_X^{01}(U) \to 0$$

when U is a coordinate disk. The surjectivity of the last map follows from the $\bar\partial$-Poincaré lemma [Griffiths-Harris, p 5]. The injectivity of the first map is clear, and the exactness in the middle from the Cauchy-Riemann equations.

There is an obvious extension of exactness for a sequence of more than 3 sheaves.

Example 2.4.4. *Again X is a Riemann surface. Then*

$$0 \to \mathbb{C}_X \to \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \mathcal{E}_X^2 \to 0$$

is exact. This can be checked on the disk, where it follows from the usual Poincaré lemma [e.g. Spivak, Calculus on manifolds]. There are couple of variants worth mentioning. We can use real valued functions and forms and everything still works. X can be replace by an n-dimensional C^∞-manifold. We still get an exact sequence as above, except that it has length n.

Let us define a functor $\Gamma : Ab(X) \to Ab$ by $\Gamma(\mathcal{F}) = \mathcal{F}(X)$.

Lemma 2.4.5. *The functor Γ is left exact, i.e. given an exact sequence*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

we have an exact sequence

$$0 \to \Gamma(\mathcal{A}) \to \Gamma(\mathcal{B}) \to \Gamma(\mathcal{C})$$

It is generally not true that the last map above is surjective, and this not just a mere technicality:

Example 2.4.6. *Let $X = \mathbb{C}^*$, then the map*

$$e : \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_X^*)$$

is not surjective because as is well known there is no way to define a holomorphic logarithm on $\mathbb{C}^*$

Following the usual pattern in homological algebra, we have

Theorem 2.4.7. *There exists a sequence of functors $H^i(X, -) : Ab(X) \to Ab$ such that*

$$H^0(X, \mathcal{F}) \cong \Gamma(\mathcal{F})$$

An exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

gives rise to a long exact sequence

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to \dots$$

We need one more fact to make this useful. The following is special case of vanishing theorem for fine sheaves. We refer to the previous references for further information.

Theorem 2.4.8. *Let X be a C^∞-manifold, and let $\mathcal{F}$ be a sheaf of C^∞_X-modules (which means that each $\mathcal{F}(U)$ is a $C^\infty(U)$-module, restrictions respect the module structure), then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

2.5 Hodge theorem in terms of sheaf cohomology

With the previous results in hand, we can do some calculations.

Proposition 2.5.1 (Dolbeault)**.** *If X is a Riemann surface, then*

$$H^1(X, \mathcal{O}_X) \cong \frac{\mathcal{E}^{01}(X)}{\bar{\partial} C^\infty(X)}$$

$$H^i(X, \mathcal{O}_X) = 0, \text{ } \textit{if } i \geq 2$$

Proof. This follows the exact sequence

$$0 \to \mathcal{O}_X \to C^\infty_X \xrightarrow{\bar\partial} \mathcal{E}^{01}_X \to 0$$

and theorems 2.4.7 and 2.4.8. □

Proposition 2.5.2 (de Rham)**.** *If X is a C^∞-manifold*

$$H^i(X, \mathbb{C}_X) \cong H^i_{dR}(X, \mathbb{C})$$

Proof. We just give the proof for $i = 1$ when X is a Riemann surface. The same method works in general. Break

$$0 \to \mathbb{C}_X \to \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \mathcal{E}^2_X \to 0$$

into exact sequences

$$0 \to \mathbb{C}_X \to \mathcal{E}^0_X \to \mathcal{E}^1_{X,cl} \to 0$$

$$0 \to \mathcal{E}^1_{X,cl} \to \mathcal{E}^1_X \xrightarrow{d} \mathcal{E}^2_X \to 0$$

Then by the above theorems

$$H^1(X, \mathbb{C}_X) = coker[H^0(X, \mathcal{E}^0_X) \to H^0(X, \mathcal{E}^1_{X,cl})]$$

$$= \frac{\ker H^0(X, \mathcal{E}^1_X) \xrightarrow{d} H^0(X, \mathcal{E}^1_X)]}{dH^0(X, \mathcal{E}^0_X)} = H^1_{dR}(X, \mathbb{C})$$

□

Theorem 2.5.3 (Hodge theorem for $\bar\partial$)**.** *If X is compact of genus g, then every element of*

$$\frac{\mathcal{E}^{01}(X)}{\bar\partial C^\infty(X)}$$

has a unique harmonic representative. Therefore

$$\dim H^1(X, \mathcal{O}_X) \cong \overline{H^0(X, \Omega^1_X)}$$

Proof. Observe that $\bar\partial$ is the $(0,1)$ part of d as well as $-i{*}d$ because

$$-i{*}df = -i({*}\partial f + {*}\bar\partial f) = -\partial f + \bar\partial f$$

Theorem 2.3.5 shows that

$$\mathcal{E}^1(X) = H \oplus dC^\infty(X) \oplus {*}dC^\infty(X)$$

where H is the space of harmonic 1-forms. Therefore the $(0,1)$-part of this decomposition yields

$$\mathcal{E}^{01}(X) = H^{01} \oplus \bar\partial C^\infty(X)$$

where H^{01} is the space of harmonic $(0,1)$-forms. This implies the theorem.

□

Corollary 2.5.4. *In the long exact sequence associated to*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to 0$$

we get an exact sequence

$$0 \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}_X) \to H^1(X, \mathcal{O}_X) \to 0$$

Proof. Since $\dim H^0(X, \Omega^1_X) = \dim H^1(X, \mathcal{O}_X) = g$ and $\dim H^1(X, \mathbb{C}_X) = 2g$, the map ι below is injective, and p is surjective

$$H^0(X, \Omega^1_X) \xrightarrow{\iota} H^1(X, \mathbb{C}_X) \xrightarrow{p} H^1(X, \mathcal{O}_X)$$

□

Remark 2.5.5. *The isomorphism*

$$\dim H^1(X, \mathcal{O}_X) \cong \overline{H^0(X, \Omega^1_X)}$$

gives a natural splitting to above projection

$$H^1(X, \mathbb{C}_X) \to H^1(X, \mathcal{O}_X)$$

Corollary 2.5.6 (Serre duality)**.** *The pairing*

$$(\alpha, \beta) = \int_X \alpha \wedge \beta$$

on $H^1(X, \mathbb{C})$ induces an isomorphism

$$H^0(X, \Omega^1_X)^* \cong H^1(X, \mathcal{O}_X)$$

Furthermore,

$$H^1(X, \Omega^1_X) \cong \mathbb{C}$$

Proof. We showed earlier that that $(\ ,\ )$ is nondegenerate. This means that given a nonzero $\alpha \in H^1(X, \mathbb{C})$, we can find $\beta \in H^1(X, \mathbb{C})$ such $(\alpha, \beta) \neq 0$. Suppose that $\alpha \in H^0(X, \Omega^1_X)$, then $(\alpha, \beta) = 0$ because $\alpha \wedge \beta = 0$. Therefore we must be able to choose $\beta \in H^1(X, \mathcal{O}_X)$ (under the decomposition explained above). Therefore the pairing induces an injection

$$H^0(X, \Omega^1_X)^* \hookrightarrow H^1(X, \mathcal{O}_X)$$

This must be an isomorphism, because the spaces have the same dimension.

By the previous corollary and proposition 2.5.1, the long exact sequence associated to

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to 0$$

gives an isomorphism

$$H^1(X, \Omega^1_X) \cong H^2(X, \mathbb{C}) \cong \mathbb{C}$$

□

2.6 Riemann's inequality

Let X be a compact Riemann surface of genus g. A function on $X - S$, where $S \subset X$ is a finite set, is called meromorphic if it holomorphic and if the Laurent expansion with respect to any coordinate has a finite number of negative terms (i.e. it has no essential singularities). Let $\mathbb{C}(X)$ denote the field of meromorphic functions on X. A basic fact, that we prove in this section, X always caries a nonconstant meromorphic function.

Given a finite set of distinct points $S = \{p_1, \ldots, p_n\}$, set $D = \sum p_i$ to the formal sum, and $\deg D = n$. If $S = \emptyset$, $D = 0$. We define a sheaf $\Omega^1_X(\log D)$ whose sections over U consist of holomorphic 1-forms on U with at worst simple poles at points of $U \cap S$.

Theorem 2.6.1 (Riemann's inequality)**.**

$$\dim H^0(X, \Omega^1_X(\log D)) \geq \deg D + g - 1$$

Proof. We define the *skyscraper* sheaf $\mathbb{C}_{p_i}$ to consist of

$$\mathbb{C}_{p_i} = \begin{cases} \mathbb{C} & \text{if } p_i \in U \\ 0 & \text{otherwise} \end{cases}$$

Then we have an exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \xrightarrow{res} \bigoplus_{p_i \in S} \mathbb{C}_{p_i} \to 0$$

where the first map is the obvious inclusion, and the second sends to form ω to the sum of its residues (defined in the usual way) at points of S. This gives rise to an exact sequenec

$$0 \to H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X(\log D)) \to \mathbb{C}^{\deg D} \to H^1(X, \Omega^1_X)$$

Since we proved that the last space is one dimensional, the theorem follows immediately. □

Corollary 2.6.2. *X has a nontrivial meromorphic function.*

Proof. By the theorem we can find 2 elements $\omega_i \in H^0(X, \Omega^1_X(\log D))$ as soon as $\deg D + g - 1 \geq 2$. Locally $\omega_= f_i(z)dz$, and the ratio $\omega_1/\omega_2 = f_1/f_2$ can be seen to a globally well defined meromorphic function. □

Note that Riemann's inequality can be improved to a much sharper statement called the *Riemann-Roch* theorem. We will not give it, since we plan to go in a different direction.