Chapter 3

The Jacobian

3.1 The Jacobian

Fix a compact Riemann surface X of genus g. Then we have seen that

$$g = \dim H^0(X, \Omega^1_X) = \frac{1}{2} rank H_1(X, \mathbb{Z})$$

Define a map to the dual space

$$H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*$$

which sends a loop γ to the integral

$$\int_\gamma \in H^0(X, \Omega^1_X)^*$$

The symbol above is the functional

$$\omega \mapsto \int_\gamma \omega$$

Proposition 3.1.1. *The image of $H_1(X, \mathbb{Z})$ in $H^0(X, \Omega^1_X)^*$ is a lattice, i.e. a subgroup generated by an $\mathbb{R}$-basis.*

Proof. Recall that we have the Hodge decomposition

$$H^1(X, \mathbb{C}) \cong H^0(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^1_X)}$$

Consider the map

$$p : H^1(X, \mathbb{R}) \to H^0(X, \Omega^1_X)$$

given by inclusion and projection. Observe that both sides have the same real dimension $2g$. If $p(\alpha) = 0$, then $\alpha = p(\alpha) + \overline{p(\alpha)} = 0$. Therefore p is injective, consequently surjective. Dually, we find that

$$H_1(X, \mathbb{R}) \cong H^0(X, \Omega^1_X)^*$$

Since $H_1(X, \mathbb{Z})$ is a lattice in $H_1(X, \mathbb{R})$, the proposition follows. □

It follows that

$$J(X) = \frac{H^0(X, \Omega^1_X)^*}{H_1(X, \mathbb{Z})}$$

is g-dimensional compex torus called the *Jacobian* of X. Fix a basis $\omega_1, \ldots, \omega_g \in H^0(X, \Omega^1_X)$, then

$$J(X) \cong \mathbb{C}^g/L$$

where

$$L = \{(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g) \mid \gamma \in H_1(X, \mathbb{Z})\}$$

is called the period lattice.

Fix a base point $x_0 \in X$, the *Abel-Jacobi* map

$$\alpha_{x_0} : X \to J(X)$$

sends x to the vector $(\int_{x_0}^x \omega_i)$. Note that we have to choose a path from x_0 to x to make sense of this. However, α_{x_0} is independent of the path, because we work mod L. It does depend on the base point, but in a fairly easy to understand way:

$$\alpha_{x_1}(x) = \alpha_{x_0}(x) - \alpha_{x_0}(x_1)$$

We usually will suppress the base point, and simply write α.

Proposition 3.1.2. *The map α is holomorphic.*

Proof. Choose a coordinate disk $U \subset X$. Since ω_i is closed, its restriction to U is df_i, for some holomorphic function f_i. Then, up to translation, $\alpha|_U = (f_1, \ldots, f_g)$ is holomorphic. □

We define

$$X^n = X \times X \times \ldots X \quad (n \text{ times})$$

and extend α to a holomorphic map $\alpha : X^n \to J(X)$

$$\alpha(x_1, \ldots, x_n) = \alpha(x_1) + \ldots \alpha(x_n)$$

Theorem 3.1.3 (Jacobi)**.** *The map $\alpha : X^g \to J(X)$ is surjective.*

Proof. The one nontrivial fact we need from several complex variables is that $\alpha(X^g)$ is an analytic subvariety [Gunning-Rossi, Analytic functions of several complex variables, p 162]. Suppose that $\alpha(X^g)$ is a proper subvariety of $J(X)$. Then it is not difficult to see that this would imply the rank of the derivative of α is strictly less that g at all points of X^g. We will derive a contradiction.

Fix a basis ω_i of $H^0(X, \Omega^1_X)$ and choose a point $(x_1, \ldots, x_g) \in X^g$. These choices will get modified as the proof proceeds. Choose coordinate disks around each x_i, and assume that $\omega_j = df_j$ in each of these disks. Then

$$\alpha(x_1, \ldots, x_g) = (f_1(x_1) + \ldots + f_1(x_g), \ldots f_g(x_1) + \ldots + f_g(x_g))$$

The derivative of α is represented by the (entirely different kind of) Jacobian

$$\left(\frac{\partial}{\partial x_j}(f_i(x_1)+\ldots)\right) = \begin{pmatrix} f_1'(x_1) & \ldots & f_1'(x_g) \\ & \ldots & \\ f_g'(x_1) & \ldots & f_g'(x_g) \end{pmatrix}$$

Since $\omega_1 \neq 0$, we can choose x_1 so that $f_1'(x_1) \neq 0$. After replacing $\omega_2, \ldots$ by $\omega_2 - c_2\omega_1$, for suitable constants c_1, we can assume that $f_2'(x_1) = f_3'(x_1)\ldots = 0$. Similarly, we can assume that $f_2'(x_2) \neq 0$. Then after a similar change of basis, we can arrange $f_3'(x_2) = \ldots 0$. Continuing this way, we can assume that the above matrix is upper triangular with nonzero entries on the diagonal. This implies that the derivative has rank g, which is the desired contradiction. □

It is possible to prove a finer statement by taking the quotient X^g/S_g, where S_g is the symmetric group. Although this space might appear to have singularities, one can show that it is a complex manfold. Then α induces a surjective holomorphic map $X^g/S_g \to J(X)$. Then on can show that

Theorem 3.1.4 (Jacobi, part II). *The map $X^g/S_g \to J(X)$ is generically one to one, i.e. there exists a dense open set $U \subset J(X)$ such that the map is an isomorphism over U.*

Example 3.1.5. *If X has genus one, then the Jacobi theorem(s) imply that $X \cong J(X)$ as complex manifolds. We earlier defined an elliptic curve to be a one dimensional complex torus. An alternative definition is that an elliptic curve is a genus one curve with a base point. The torus structure then follows from the above isomorphism.*

3.2 Divisor class group

Let X be compact Riemann surface as before. The group $Div(X)$ of divisors on X is the free abelian group. The elements, called *divisors* are finite formal sums $D = \sum_i n_i p_i$ with $p_i \in X$. The degree $\deg D = \sum_i n_i$. Set $Div^0(X) = \ker : Div(X) \xrightarrow{\deg} \mathbb{Z}$. Natural examples arise from meromorphic functions. Let $\mathbb{C}(X)$ denote the set of meromorphic functions on X. This is a field. Given $x \in X$, choose coordinate z at x. Given $f \in \mathbb{C}(X)$, we can expand it as Laurent series

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$$

If $a_n \neq 0$, set $ord_x(f) = n$. Define $ord_x(0) = \infty$. Then

Proposition 3.2.1. *The function $ord_x : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$ is a discrete valuation, i.e.*

1. *If $f, g \in \mathbb{C}(X)^*$, $ord_x(fg) = ord_x(f) + ord_x(g)$*
2. *$ord_x(f+g) \geq ord_x(f) + ord_x(g)$*

Proof. Elementary. □

Given $f \in \mathbb{C}(X)^*$, set

$$div(f) = \sum_{x \in X} ord_x(f)x$$

Since the zeros and poles of f are isolated and X is compact, this sum is finite, so it defines a divisor. A divisor of this form is called *principal*.

Lemma 3.2.2. *The map $div : \mathbb{C}(X)^* \to Div(X)$ is a homomorphism. Therefore the image $Princ(X)$ is a subgroup.*

Proof. This is immediate from the previous proposition. □

The *divisor class group*

$$Cl(X) = \frac{Div(X)}{Princ(X)}$$

Theorem 3.2.3. *The degree of a principal divisor is* 0.

Proof. Given $f \in \mathbb{C}(X)^*$, choose coordinate disks D_j around each zero or pole of f. Then by basic complex analysis

$$\deg div(f) = \sum_j \frac{1}{2\pi i} \int_{\partial \overline{D}_j} \frac{f'(z)}{f(z)} dz$$

By the Stokes' theorem, the integral on the right equals

$$\frac{1}{2\pi i} \int_{X - \cup D_j} d\left(\frac{f'(z)}{f(z)} dz\right) = 0$$

□

Therefore $Princ(X) \subset Div^0(X)$, and we define

$$Cl^0(X) = \frac{Div^0(X)}{Princ(X)}$$

It should be clear that there is an exact sequence

$$0 \to Cl^0(X) \to Cl(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

so that $Cl^0(X)$ is the interesting part of the class group.

Exercise 3.2.4. *Check that $X = \mathbb{P}^1$, any degree zero divisor is the divisor of a rational function. It follows that $Cl^0(X) = 0$.*

The next theorem explains a fundamental property of the Jacobian.

Theorem 3.2.5 (Abel-Jacobi). *If X is a compact Riemann surface, then there is an isomorphism*

$$Cl^0(X) \cong J(X)$$

as abstract groups.

We will give a proof later on, but we want to make some comments about it now. We define homomorphism

$$\alpha : Div(X) \to J(X)$$

by

$$\alpha(\sum_i n_i x_i) = \sum_i n_i \alpha(x_i)$$

Jacobi's theorem implies that the restriction $Div^0(X) \to J(X)$ is surjective. The other half of the above theorem is

Theorem 3.2.6 (Abel). *The kernel of $\alpha : Div^0(X) \to J(X)$ is $Princ(X)$.*

Example 3.2.7. *If $X = \mathbb{P}^1$, then we recover that $Cl^0(X) = J(X) = 0$.*

Example 3.2.8. *If X is an elliptic curve then $Cl^0(X) \cong X$.*

3.3 Dual description of the Jacobian

Let X be compact Riemann surface of genus g. The long exact sequence associated the exponential sequence is

$$0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^*) \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$$

The first few terms are simply

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{e} \mathbb{C}^*$$

Since the last map is surjective, the next few terms of the previous sequence becomes

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to J^\vee(X) \to 0$$

where we define

$$J^\vee(X) = \ker : H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$$

Theorem 3.3.1. *We have an isomorphism*

$$J(X) \cong J^\vee(X)$$

Sketch. By the Poincaré and Serre duality theorems, we have isomorphisms labelled P and S below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(X,\mathbb{Z}) & \longrightarrow & H^0(X,\Omega^1_X)^* & \longrightarrow & J(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & J^\vee(X) & \longrightarrow & 0 \end{array}$$

Therefore the desired isomorphism follows. □

We will say more about this isomorphism when we discuss abelian varieties.

3.4 Čech cohomology and Abel's theorem

Before giving the proof of Abel's theorem, we need to make a digression into homological algebra. Given an open cover $\mathcal{U} = \{U_i\}$ of a space X, let $U_{ij} = U_i \cap U_j$ etc. A Čech 1-*cocycle* with coefficients in a sheaf $\mathcal{F}$ with respect to $\mathcal{U}$ is a collection $f_{ij} \in \mathcal{F}(U_{ij})$ such that

$$f_{ij} = -f_{ji}$$

and the restrictions to U_{ijk} satisfy

$$f_{ij} + f_{jk} + f_{ki} = 0$$

Let $Z^1(\mathcal{U}, \mathcal{F})$ denote the group of 1-cocycles. We define f_{ij} to be a 1-*coboundary* if there exists $\phi_i \in \mathcal{F}(U_i)$ such that

$$f_{ij} = \phi_i - \phi_j$$

Let $B^1(\mathcal{U}, \mathcal{F})$ denote the subgroup of 1-coboundaries . Define the Čech cohomology groups

$$\check{H}^1(\mathcal{U}, \mathcal{F}) = \frac{Z^1(\mathcal{U}, \mathcal{F})}{B^1(\mathcal{U}, \mathcal{F})}$$

$$\check{H}^1(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{F})$$

Theorem 3.4.1. *We have an isomorphism*

$$H^1(X, \mathcal{F}) \cong \check{H}^1(X, \mathcal{F})$$

If $\mathcal{U}$ is an open cover such that $H^i(U_{j_1 \ldots j_k}, \mathcal{F}) = 0$ for all $i > 0$ and all indices $j_1, \ldots,$ in which the cover is called Leray, then

$$H^i(X, \mathcal{F}) \cong \check{H}^1(\mathcal{U}, \mathcal{F})$$

Let us return to the setting where X is a compact Riemann surface of genus g. Suppose that $D = n_1x_1 + n_2x_2 \ldots$ is a nontrivial divisor. Construct and open cover $\mathcal{U} = \{U_0, U_1 \ldots\}$ where $U_0 = X - \{x_1, x_2, \ldots\}$ and $U_1, \ldots$ are coordinate disks around $x_1, \ldots$ chosen small enough that none of disks intersect. Let $z_1, \ldots$ denote coordinates. If $j > 0$ set $\phi_{0j} = z_j^{n_j} \in \mathcal{O}(U_0 \cap U_j)$, and $\phi_{j0} = z_j^{-n_j}$, since there are no triple intersections, we can see that this This forms a 1-cocycle on $\mathcal{U}$ with values in $\mathcal{O}_X^*$. Let $[D]$ denote the class of this cocycle in $H^1(X, \mathcal{O}_X^*)$. If D is trivial, then we simply define $[D] = 0$.

Theorem 3.4.2. *If* $\deg D = 0$*, then* $[D] \in J^\vee(X)$*. Under the previous isomorphism, this coincides up to sign with* $\alpha(D)$*.*

Abel's theorem now becomes the following statement.

Theorem 3.4.3 (Abel)**.** *If* $D \in Div^0(X)$*, then* $[D] = 0$ *if and only of* $D \in Princ(X)$*.*

Proof. Let $n_j, \{U_j\}, \phi_{ij}$ be as above. Suppose that $D = div(f)$. Set $\phi_0 = f|_{U_0}$ and $\phi_j = (f|_{U_j}/z_j^{n_j})$ for $j > 0$. Then

$$\phi_{ij} = \phi_i \phi_j^{-1}$$

This means that ϕ_{ij} is a coboundary. Therefore $[D] = 0$.

Suppose $[D] = 0$. We can assume that $D \neq 0$. Then the cover $\{U_j\}$ consists of noncompact open sets. The only nontrivial fact we assume without proof is that the open cover is a Leray cover[1] for $\mathcal{O}_X^*$. Therefore $H^1(X, \mathcal{O}_X^*) = \check{H}^1(\{U_j\}, \mathcal{O}_X^*)$. It follows that ϕ_{ij} is a coboundary so that

$$\phi_{ij} = \phi_i \phi_j^{-1}$$

for some appropriate functions on the right. Set $f = \phi_0$, and we can see that $D = div(f)$.

□

3.5 Riemann bilinear relations

Let X be a compact Riemann surface of genus g. Recall

$$(\alpha, \beta) = \int_X \alpha \wedge \beta$$

defines a nondegenerate skew symmetric blinear form on $H^1_{dR}(X, \mathbb{C})$. This corresponds to the interesction form on $H_1(X, \mathbb{Z})$ under the de Rham isomorphism

$$H^1_{dR}(X, \mathbb{C}) = Hom(H_1(X, \mathbb{Z}), \mathbb{C})$$

[1] This follows from Poincaré duality which implies that $H^2(U_{j_1,\ldots}, \mathbb{Z}) = 0$ and theorem of Benke-Stein that $U_{j_1,\ldots}$ are Stein which implies $H^1(U_{j_1,\ldots}, \mathcal{O}_X) = 0$

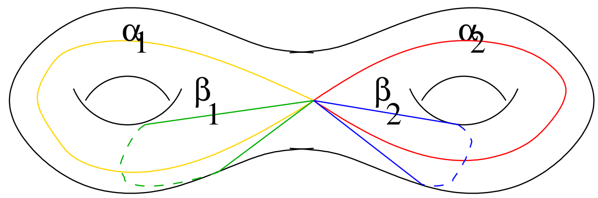
A basis $\alpha_1, \ldots, \alpha_g, \beta_1 \ldots \beta_g \in H_1(X, \mathbb{Z})$ is called symplectic if

$$(\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0$$

and

$$(\alpha_i, \beta_j) = \delta_{ij}$$

Such bases exist, for example a standard choice is



If $\alpha_1^*, \ldots \beta_g^* \in H^1(X, \mathbb{Z})$ denotes the dual basis, then they will also satisfy the same symplectic relations.

Theorem 3.5.1 (Riemann bilinear relations)**.** *Choose* $\omega, \eta \in H^0(X, \Omega_X^1)$ *and a symplectic basis* $\alpha_1, \ldots, \alpha_g, \beta_1 \ldots \beta_g \in H_1(X, \mathbb{Z})$*. Then*

(a)

$$(\omega, \eta) = 0$$

(b)

$$\sum_k \left(\int_{\alpha_k} \omega \int_{\beta_k} \eta - \int_{\beta_k} \omega \int_{\alpha_k} \eta \right) = 0$$

(c) If $\omega \neq 0$*, then*

$$i(\omega, \bar{\omega}) > 0$$

(d) If $\omega \neq 0$*, then*

$$Im \left(\sum_k \overline{\int_{\alpha_k} \omega} \int_{\beta_k} \omega \right) > 0$$

Proof. For (a) observe that $\omega \wedge \eta = 0$. Item (b) follows from (a) by expanding

$$\omega = \sum_k \left(\int_{\alpha_k} \omega \, \alpha_k^* + \int_{\beta_k} \omega \, \beta_k^* \right)$$

$$\eta = \sum_k \left(\int_{\alpha_k} \eta \, \alpha_k^* + \int_{\beta_k} \eta \, \beta_k^* \right)$$

and using the symplectic relations. For (c) write $\omega = f(z)dz$, then

$$(\omega, \bar{\omega}) = 2 \int_X |f(z)|^2 dx \wedge dy > 0$$

Item (d) follows from (c) by expanding ω in $\alpha_1^*, \ldots, \beta_g^*$ as above. $\square$

Corollary 3.5.2. *A basis $\omega_1, \dots, \omega_g \in H^0(X, \Omega^1_X)$ can be chosen so that the $g \times g$ matrix*

$$\left(\int_{\alpha_j} \omega_i\right) = I$$

and

$$\Omega = \left(\int_{\beta_j} \omega_i\right)$$

is symmetric with positive definite imaginary part.

Proof. Choose any basis ω_i and let

$$A = \left(\int_{\alpha_j} \omega_i\right), \quad B = \left(\int_{\beta_j} \omega_i\right)$$

Then (b) implies that AB^T is symmetric, and (d) implies that AB^T has positive definite imaginary part. Therefore AB^T is invertible, so the same is true for A. Consequently, we can change basis so that $(A, B) \mapsto (A^{-1}A, A^{-1}B) = (I, \Omega)$. Repeating the previous argument shows that Ω is symmetric with positive definite imaginary part. □

We say that the basis ω_i, or period matrix, is normalized if it satisfies the conditions of the previous corollary. We define the $g \times g$ *Siegel upper half plane* to be

$$\mathbb{H}_g = \{\Omega \in Mat_{g\times g} \mid \Omega^T = \Omega, Im\,\Omega > 0\}$$

To summarize what we did in this section, X plus a choice of symplectic basis for $H_1(X, \mathbb{Z})$ determines a point of $\mathbb{H}_g$.